Symmetry and Transformation Properties of Linear Iterative Ordinary Differential Equation

Mensah Kékéli Folly-Gbetoula (Mensah.Folly-Gbetoula@students.wits.ac.za)

University of the Witwatersrand, Johannesburg

School of Mathematics

Supervisor: Professor A.H. Kara, University of the Witwatersrand, South Africa

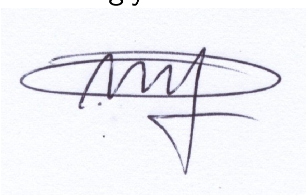
Co-Supervisor: Dr J.C. Ndogmo, University of the Witwatersrand, South Africa

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of science.

Johannesburg, December 2012.

Declaration

I, the undersigned, hereby declare that the work contained in this Dissertation is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Mensah Kékéli Folly-Gbetoula, today

Abstract

Solutions of linear iterative equations and expressions for these solutions in terms of the parameters of the source equation are obtained. Based on certain properties of iterative equations, finding the solutions is reduced to finding group-invariant solutions of the second-order source equation. We have therefore found classes of solutions to the source equations. Regarding the expressions of the solutions in terms of the parameters of the source equation, an ansatz is made on the original parameters r and s, by letting them be functions of a specific type such as monomials, functions of exponential and logarithmic type. We have also obtained an expression for the source parameters of the transformed equation under equivalence transformations and we have looked for the conservation laws of the source equation. We conducted this work with a special emphasis on second-, third- and fourth-order equations, although some of our results are valid for equations of a general order.

Acknowledgements

Firstly, I thank God who has helped me to finish the Master's program. My sincere gratitude goes to my supervisors, Prof Abdul Kara and Dr Jean-Claude Ndogmo for their help. Thanks to my family and friends for their encouragement. Lastly, I would like to thank the African Institute for Mathematical Sciences and the research centre of the university of the Witwatersrand for their financial support.

Contents

1. Introduction

Linear iterative equations are the iterations of a linear first-order equation. They are known as equations that can always be reduced to the canonical form $y^{(n)} = 0$ by point transformations. It is well known that every second-order linear ordinary differential equation can be reduced to the canonical form $y'' = 0$ by an invertible point transformation. However, the corresponding property does not hold for equations of order higher than two and any equation of such an order can be transformed into the canonical form if and only if it is iterative [1]. On the basis of this and a result of S. Lie [2], iterative equations are also the only linear equations that admit a symmetry algebra of maximal dimension. Moreover, the general solution of iterative equations of a general order can be obtained by a very simple superposition formula from those of the source equation of the second-order.

1.1 Overview about iterative equations

Linear ordinary differential equations of a general order have been studied in the recent literature and from the symmetry group approach by many authors [1, 3, 4, 13]. It is well-known that for the order $n = 2$, the dimension of the symmetry algebra does not exceed 8 and all linear differential equations are locally equivalent to the canonical form $y'' = 0$. For $n \geq 3$, Sophus Lie proved that the dimension of the symmetry algebra does not exceed $n + 4$. One of Lie's main results is that the maximal dimension is reached for equations reducible to the canonical form $y^{(n)} = 0$.

In their work, Krause and Michel [1] proved that an equation is reducible to the canonical form if its symmetry algebra has maximal dimension. Then, using the result due to Lie cited above, they showed that for a linear equation of order $n \geq 3$ the statements

(a) the equation is reducible to the form $y^{(n)} = 0$ by a diffeomorphism of the (x, y)-plane

(b) the Lie algebra of its symmetry group has maximal dimension

(c) the equation is iterative

are equivalent. By definition, iterative equations are the iterations

$$\Psi y \equiv r(x)y' + s(x)y = 0 \tag{1.1a}$$

$$\Psi^n y \equiv \Psi^{(n-1)} [\Psi y], \qquad n \in \mathbb{N}, \tag{1.1b}$$

where $\Psi = r\frac{d}{dx} + s$ is a differential operator and r and s are given functions of x referred to as the parameters of the source equation $\Psi y \equiv r(x)y' + s(x)y = 0$.

Let us consider a linear differential equation of a general order n in its standard form

$$y^{(n)} + \sum_{i=0}^{n-1} b_i\, y^{(i)} = 0, \tag{1.2}$$

where the b_i are functions of the independent variable x. For $n = 3$, Lie [2] and Laguerre [14] showed that the equation is reducible to the form $y^{(n)} = 0$, which we shall refer to as the canonical form, if and only if the coefficients in (1.2) satisfy

$$54b_0 - 18b_1b_2 + 4b_2^3 - 27b_1' + 18b_2b_2' + 9b_2'' = 0. \tag{1.3}$$

It is well-known that one can use the transformation

$$y \mapsto y \exp\left(\frac{1}{n}\int_{x_o}^{x} b_{n-1}(v)dv\right) \tag{1.4}$$

to reduce the general form (1.2) into the reduced normal form

$$y^{(n)} + \sum_{i=0}^{n-2} a_i\, y^{(i)} = 0, \tag{1.5}$$

and in the case of iterative equations, the operator that generates an iterative equation of a general order n in its normal form (1.5) has been found [4]. We know that up to isomorphism the symmetry algebra of a differential equation does not change under an invertible point transformation, meaning that (1.2) and (1.5) have isomorphic symmetry algebras. Therefore, for several considerations we may without loss of generality let the iterative equation be on the form (1.5).

Some properties of iterative equations were obtained and the characterizations of these equations in terms of their coefficients have been considered [4, 13]. All the coefficients a_i can naturally be expressed in terms of the parameters r and s of the source equation [4] but surprisingly it is always possible to express the coefficients a_{n-i} for $2 < i \leq n$ in terms of the coefficient a_{n-2} and its derivatives [13]. The list of iterative equations in which all the coefficients are given in terms of a_{n-2} and its derivatives for n running between 3 and 8 was obtained in [13]. The first three of them are

$$y^{(3)} + a_1y^{(1)} + \frac{1}{2}a_1^{(1)}y = 0 \tag{1.6}$$

$$y^{(4)} + a_2y^{(2)} + a_2^{(1)}y^{(1)} + (\frac{3}{10}a_2^{(2)} + \frac{9}{100}a_2^2)y = 0 \tag{1.7}$$

$$y^{(5)} + a_3y^{(3)} + \frac{3}{2}a_3'y^{(2)} + (\frac{9}{10}a_3'' + \frac{16}{100}a_3^2)y' + (\frac{1}{5}a_3^{(3)} + \frac{16}{100}a_3a_3')y = 0 \tag{1.8}$$

It should be noted that if we let

$$y^{(n)} + A_n^2y^{(n+2)} + \cdots + A_n^{n-1}y^{(1)} + A_n^n y = 0 \tag{1.9}$$

be the general form of linear iterative equations in normal form for the same source equation, then by a result of [4] we have

$$A_n^2 = \binom{n+1}{3}A_2^2. \tag{1.10}$$

Using the result from [13] one can generate the list of canonical forms of iterative equations in normal reduced form for any order after a long and sometimes very complicated set of calculations.

Another exceptional property of iterative equation states that if we assume that u and v are the independent solutions of the second-order source equation

$$y'' + p(x)y = 0, \tag{1.11}$$

where p turns out to be the Wronskian of u and v, then n linearly independent solutions of (1.5) are given by [1]

$$y_k = u^{n-(k+1)}v^k \qquad 0 \leq k \leq n-1. \tag{1.12}$$

Therefore, once we know the general solution of the source equation (1.11) we can construct the set of solutions to the corresponding nth-order iterative equation. The implication is that there is no need to search for linearly independent solutions for the linear differential equation of order n itself when we know those of its source equation. In other words, finding the general solution of the nth-order equation (1.5) is equivalent to finding the two linearly independent solutions of the second-order source equation (1.11).

1.2 Coefficients of the iterative equation in terms of the coefficient A_2^2

To rewrite the coefficients of the linear iterative equation in terms of the coefficient A_2^2 of the second-order source equation and its derivatives only, let

$$y^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0 \tag{1.13}$$

be a linear iterative equation in normal form, where A_2^2 are functions of x, and let

$$y'' + A_2^2(x)y = 0 \tag{1.14}$$

be the corresponding second-order source equation.
If we assume that the first-order source equation in standard form is

$$r(x)y' + s(x)y = 0 \tag{1.15}$$

where $r = r(x)$ and $s = s(x)$ are the parameters of the source equations, it follows [4] that

$$A_2^2(x) = \frac{r'^2 - 2rr''}{4r^2} \tag{1.16a}$$

provided that

$$s = -\frac{1}{2}(n-1)r'. \tag{1.16b}$$

It is well-known that it is always possible to express the coefficients of the iterative equation (1.13) in terms of A_n^2 [13]. On the other hand, it has been proved [4] that the relationship between A_n^2 and A_2^2 is given by

$$A_n^2 = \binom{n+1}{3} A_2^2. \tag{1.17}$$

It follows from these results that all the coefficients of iterative equations with the same source equation can be written in terms of the coefficient A_2^2 of the second-order source equation.
Below is the list of iterative equations of order $n = 3, 4, 5$ involving the coefficient A_2^2, a say, of the second-order source equation and its derivatives only.

$$y^{(3)} + 4ay^{(1)} + 2a'y = 0 \tag{1.18}$$

$$y^{(4)} + 10ay^{(2)} + 10a'y^{(1)} + (3a'' + 9a^2)y = 0 \tag{1.19}$$

$$y^{(5)} + 20ay^{(3)} + 30a'y^{(2)} + \left(18a'' + 64a^2\right) y^{(1)} + \left(4a^{(3)} + 64aa'\right) y = 0 \tag{1.20}$$

$$\begin{aligned} &y^{(6)} + 35ay^{(4)} + 70a'y^{(3)} + \left(63a^{(2)} + 259a^2\right) y^{(2)} + \left(28a^{(3)} + 518aa'\right) y' \\ &+ \left(5a^{(4)} + 130a'^2 + 155aa^{(2)} + 225a^3\right) y = 0. \end{aligned} \tag{1.21}$$

The list can be extended to a general order, although the general formula is not known.

2. Lie analysis and Group-invariant solutions

2.1 Introduction

In this chapter, some background to Lie symmetry analysis and the mathematical methods to generate the Lie symmetries of differential equations are provided. The method of symmetry analysis is a useful tool for finding exact solutions to a differential equation via the invariance of the equation under group transformations or via the similarity variables.

2.2 Definitions

Let us start the definitions with the concept of a group.

2.2.1 Definition. A **group** is a set G together with a group operation, usually called multiplication, such that for any two elements g and h of G, the product $g \cdot h$ is again an element of G. The group operation is required to satisfy the following axioms:

- *Associativity*. For any elements g, h, k of G:
$$g \cdot (h \cdot k) = (g \cdot h) \cdot k. \tag{2.1}$$
- *Identity Element*. There exists a unique element e of G, called the identity element, which has the property that
$$e \cdot g = g = g \cdot e \tag{2.2}$$
for all g in G.
- *Inverses*. For each element g in G, there exists a unique inverse element, denoted g^{-1} in G, with the property
$$g \cdot g^{-1} = e = g^{-1} \cdot g. \tag{2.3}$$

2.2.2 Definition. An r **parameter Lie group** is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operator

$$m : G \times G \to G, \qquad m(g, h) = g \cdot h, \qquad g, h \in G, \tag{2.4}$$

and the inversion

$$i : G \to G, \qquad i(g) = g^{-1}, \qquad g \in G, \tag{2.5}$$

are smooth maps between manifolds.

2.2.3 Definition. A **Lie subgroup** H of a Lie group G is given by a submanifold $\phi : \tilde{H} \to G$, where $\tilde{H}$ itself is a Lie group, $H = \phi(\tilde{H})$ is the image of ϕ, and ϕ is a Lie group homomorphism.

To understand symmetry we require the knowledge of group transformations.

2.2.4 Definition. Let M be a smooth manifold. A **local group of transformations** acting on M is given by a (local) Lie group G, an open subset $\mathcal{U}$, with

$$\{e\} \times M \subset \mathcal{U} \subset G \times M, \tag{2.6}$$

which is the domain of definition of the group action, and a smooth map $\Psi : \mathcal{U} \to M$ with the following properties:

- If $(h, x) \in \mathcal{U}, (g, \Psi(h, x)) \in \mathcal{U}$, and also $(g \cdot h, x) \in \mathcal{U}$, then
$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x). \tag{2.7}$$
- For all $x \in M$,
$$\Psi(e, x) = x. \tag{2.8}$$
- If $(g, x) \in \mathcal{U}$, then $(g^{-1}, \Psi(g, x)) \in \mathcal{U}$ and
$$\Psi(g^{-1}, \Psi(g, x)) = x. \tag{2.9}$$

2.2.5 Definition. Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-*invariant*, and G is called a **symmetry group** of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.

2.2.6 Definition. A **Lie algebra** is a vector space L over a field F (that we shall assume to be of characteristic zero) with a bilinear bracket operation (the commutator)

$$[.,.] : L \times L \to L$$

which satisfies the axioms

- *Bilinearity*
$$[cv + c'v', w] = c[v, w] + c'[v', w], \quad [v, cw + c'w'] = c[v, w] + c'[v, w'],$$
for c, $c' \in F$,
- *Skew-Symmetry*
$$[v, w] = -[w, v].$$
- *Jacobi identity*
$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0,$$
for all $u, v, v', w, w' \in L$.

In this work F will assumed to be the field of real number.

2.3 Lie analysis

The symmetry group of a system of differential equations is the largest group of point transformations acting on the space of dependent and independent variables that leaves the equation invariant. Let

$$x^\star = X(x;\varepsilon) \tag{2.10}$$

be a one parameter Lie group of transformations.

2.3.1 Definition. The **infinitesimal generator** of the one-parameter Lie group of transformations $x^\star = X(x;\varepsilon)$ is the operator

$$X = X(x) = \xi(x) \times \nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}, \tag{2.11}$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \dots, \frac{\partial}{\partial x_n}\right). \tag{2.12}$$

2.3.2 Definition. An infinitely differentiable function F is **invariant function** of the Lie group of transformations (2.10) if and only if, for any group transformations,

$$F(x^\star) = F(x). \tag{2.13}$$

2.3.3 Theorem. $F(x)$ *is invariant under the Lie group of transformations* (2.10) *if and only if,*

$$XF(x) = 0. \tag{2.14}$$

Consider a system of nth-order differential equations

$$\Delta_\mu(x, y^{(n)}) = 0, \quad \mu = 1, \dots, l, \tag{2.15}$$

involving p independent variables $x = (x^1, \dots, x^p)$, q dependent variables $y = (y^1, \dots, y^q)$ and the derivatives of y with respect to x up to order n. We search for a one-parameter Lie group of point transformations $\Psi \equiv (x^\star, y^\star) = (\Psi_1, \Psi_2)$ of this equations of the form

$$x^\star = x + \varepsilon\xi(x, y) \equiv \Psi_1(\varepsilon, x, y) \tag{2.16}$$
$$y^\star = y + \varepsilon\phi(x, y) \equiv \Psi_2(\varepsilon, x, y) \tag{2.17}$$

where ε is the group parameter with a symmetry generator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, y)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \Phi_\alpha(x, y)\frac{\partial}{\partial y^\alpha}. \tag{2.18}$$

The integration of the initial value problem

$$\mathbf{v}\Big|_{\Psi(\varepsilon,x,y)} = \frac{d}{d\varepsilon}\Psi(\varepsilon,x,y) \tag{2.19a}$$

$$\Psi(0,x,y) = (x,y) \tag{2.19b}$$

determines the group transformations Ψ. In other words,

$$\left.\frac{dx^{i\star}}{d\varepsilon}\right|_{\varepsilon=0} = \xi^i(x,y), \qquad i=1,\dots,p, \tag{2.20a}$$

and

$$\left.\frac{dy^{\alpha\star}}{d\varepsilon}\right|_{\varepsilon=0} = \phi_\alpha(x,y), \qquad \alpha=1,\dots,q. \tag{2.20b}$$

For an nth-order differential equations, we require the knowledge of the nth extension of $\mathbf{v}$. The following theorem gives the general prolongation formula $\mathrm{pr}^{(n)}\mathbf{v}$ of the infinitesimal generator $\mathbf{v}$.

2.3.4 Theorem. *Let*

$$\mathbf{v} = \sum_{i=1}^{p}\xi^i(x,y)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\Phi_\alpha(x,y)\frac{\partial}{\partial y^\alpha} \tag{2.21}$$

be a vector field defined on an open subset $M \subset X \times U$. The nth prolongation of $\mathbf{v}$ is the vector field

$$pr^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J}\Phi_\alpha^J(x,y)\frac{\partial}{\partial y_J^\alpha} \tag{2.22}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, j_2, \dots, j_k)$, with $1 \le j_k \le p, \quad 1 \le k \le n$. The coefficient functions Φ_α^J of $pr^{(n)}\mathbf{v}$ are given by the following formula:

$$\Phi_\alpha^J(x,y) = D_J\left(\Phi_\alpha - \sum_{i=1}^{p}\xi^i y_i^\alpha\right) + \sum_{i=1}^{p}\xi^i y_{J,i}^\alpha, \tag{2.23}$$

where $y_i^\alpha = \partial y^\alpha/\partial x^i$, and $y_{J,i}^\alpha = \partial y_J^\alpha/\partial x^i$.

The infinitesimal criterion for invariance is given by the following theorem.

2.3.5 Theorem. *Suppose*

$$\Delta_\mu(x, y^{(n)}) = 0, \quad \mu = 1,\dots,l, \tag{2.24}$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}\mathbf{v}[\Delta_\mu] = 0, \quad \mu = 1,\dots,l, \quad \text{whenever} \quad \Delta_\mu(x,y^{(n)}) = 0, \tag{2.25}$$

for every infinitesimal generator $\mathbf{v}$ of G, then G is a symmetry group of the system.

For an nth-order ordinary differential equation (ODE) we have up to nth derivatives and so need an nth extension so that we can investigate how the derivatives transform too. Therefore, If we assume that $E = 0$ is an nth-order ODE then the invariance criterion is given by

$$\mathbf{v}^{[n]}E = 0 \quad \text{whenever} \quad E = 0, \tag{2.26}$$

where $\mathbf{v}^{[n]}$ stands for $\text{pr}^{(n)}\mathbf{v}$, that is,

$$\mathbf{v}^{[n]} = \xi\partial_x + \phi\partial_y + \phi^x\partial_{y_x} + \phi^{xx}\partial_{y_{xx}} + \phi^{xxx}\partial_{y_{xxx}} + \cdots + \phi^{[n]}\partial_{y^n}. \tag{2.27}$$

Here,

$$\begin{aligned}\phi^x &= D_x(\phi - \xi y_x) + \xi y_{xx}\\ &= D_x\phi - D_x(\xi y_x) + \xi y_{xx}\\ &= \phi_x + \phi_y y_x - \xi_x y_x - \xi_y {y_x}^2,\end{aligned} \tag{2.28a}$$

$$\begin{aligned}\phi^{xx} &= D_{xx}(\phi - \xi y_x) + \xi y_{xxx}\\ &= \phi_{xx} + (2\phi_{xy} - \xi_{xx})y_x + (\phi_{yy} - 2\xi_{xy}){y_x}^2 + (\phi_y - 2\xi_x)y_{xx}\\ &\quad - \xi_{yy}y_x^3 - 3\xi_y y_x y_{xx},\end{aligned} \tag{2.28b}$$

$$\begin{aligned}\phi^{xxx} =& D_{xxx}(\phi - \xi y_x) + \xi y_{xxxx}\\ =& \phi_{xxx} + (3\phi_{xxy} - \xi_{xxx})y_x + (3\phi_{xyy} - 3\xi_{xxy}){y_x}^2\\ &+ (3\phi_{xy} - 3\xi_{xx})y_{xx} \quad + (\phi_{yyy} - 3\xi_{yyx})y_x^3 + (3\phi_{yy} - 9\xi_{xy})y_x y_{xx}\\ &+ (\phi_y - 3\xi_x)y_{xxx} \quad - \xi_{yyy}y_x^4 - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2 - 4\xi_y y_x y_{xxx},\end{aligned} \tag{2.28c}$$

$$\begin{aligned}\phi^{xxxx} &= D_{xxxx}(\phi - \xi y_x) + \xi y_{xxxxx}\\ &= \phi_{xxxx} + (4\phi_{xxxy} - \xi_{xxxx})y_x + (6\phi_{xxyy} - 4\xi_{xxxy}){y_x}^2\\ &\quad + (4\phi_{xyyy} - 6\xi_{xxyy})y_x^3 + (\phi_{yyyy} - 4\xi_{yyyx})y_x^4 - \xi_{yyyy}y_x^5\\ &\quad + (6\phi_{xxy} - 4\xi_{xxx})y_{xx} + (12\phi_{xyy} - 18\xi_{xxy})y_x y_{xx} +\\ &\quad [6\phi_{yyy} - 24\xi_{xyy}]y_x^2 y_{xx} + [3\phi_{yy} - 12\xi_{xy}]y_{xx}^2 + [4\phi xy - 6\xi_{xx}]y_{xxx}\\ &\quad + [4\phi_{yy} - 16\xi_{xy}]y_x y_{xxx} - 10\xi_y y_{xx} y_{xxx} - 10\xi_{yy}y_x^2 y_{xxx}\\ &\quad + [\phi_y - 4\xi_x]y_{xxxx} - 5\xi_y y_x y_{xxxx} - 10\xi_{yyy}y_x^3 y_{xx} - 15\xi_{yy}y_x y_{xx}^2.\end{aligned} \tag{2.28d}$$

and [1]

$$\begin{aligned}\phi^{[n]} =& -[(n\xi_x - \phi_y)]\, y^{(n)} - [(n+1)\xi_y]\, y' y^{(n)}\\ &- \binom{n+1}{2}\xi_y y'' y^{(n-1)} + n(\phi_{yy} - n\xi_{xy})y' y^{(n-1)} + n(\phi_{xy}\\ &- \frac{n-1}{2}\xi_{xx})y^{(n-1)} + \frac{n}{2}\left(\phi_{xxy} - \frac{n-2}{3}\xi xxx\right) y^{(n-2)} + \ldots .\end{aligned} \tag{2.28e}$$

Theorem 2.3.5 leads to a nonlinear partial differential equation in ξ and ϕ. We then equate all the coefficients of all powers of derivatives of y to zero because ξ and ϕ depend only on x and y. The system of determining equations obtained gives the expression of ξ and ϕ. Note that the number of constants found determines the dimension of the Lie group.

2.4 Group-invariant solutions

Although the full symmetry group of a differential equation transforms each solution to another solution, for certain subgroups H of G, there are solutions which are transformed into themselves. Such solutions considered as a set are thus locally H-invariant sets, and are termed group-invariant solutions. They can be found for ordinary differential equations by solving an equation of lower order in which the new variables are the differential invariants of H.

2.4.1 Definition. A **regular submanifold** N of a manifold M is a submanifold parametrized by $\phi : \tilde{N} \to M$ with the property that for each x in N there exist arbitrarily small open neighbourhoods U of x in M such that $\phi^{-1}[U \cap N]$ is a connected open subset of $\tilde{N}$.

2.4.2 Proposition. Let G act regularly on $M \subset X \times U$. Then $z_0 \in M$ lies in $I^{(0)}$ ($I^{(0)} \subset M$ consists of all points $z_0 = (x_0, u_0)$ such that there is at least one locally G-invariant function $u = f(x)$ whose graph passes through z_0) if and only if the orbit of G through z_0 is transverse to the vertical space U_{z_0}, in which case G is said to act transversally at z_0.

2.4.3 Theorem. *Let G act regularly and transversally on $M \subset X \times U$. Let*

$$\mathbf{v}_k = \sum_{i=1}^{p} \xi_k^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha^k(x,u)\frac{\partial}{\partial u^\alpha}, \qquad k = 1, \ldots, r, \tag{2.29}$$

be a basis for the infinitesimal generators. Then the nth invariant space $I^{(n)} \subset M^{(n)}$ is determined by the equations

$$I^{(n)} = \{(x, u^{(n)}) : D_J Q_\alpha^k(x, u^{(n)}) = 0,\, k = 1, \ldots, r,\, \alpha = 1, \ldots, q,\, \#J \le n-1\}, \tag{2.30}$$

where $Q_\alpha^k = \phi_\alpha^k - \sum_i \xi_k^i u_i^\alpha$ are the characteristics of the vector fields $\mathbf{v}_k$.

2.4.4 Proposition. Let G be a group of transformations acting on $M \subset X \times U \simeq \mathbb{R}^2$. Suppose $y = \eta(x, u^{(n)})$ and $w = \zeta(x, u^{(n)})$ are nth-order differential invariants of G. Then the derivative

$$\frac{dw}{dy} = \frac{dw/dx}{dy/dx} \equiv \frac{D_x\zeta}{D_x\eta} \tag{2.31}$$

is an (n+1)-st order differential invariant for G.

2.4.5 Example. Consider the iterative equation of order three in its standard form

$$u^{(3)} + 3u^{(2)} + 3u^{(1)} + u = 0. \tag{2.32}$$

It can be verified that one of its symmetries is $\mathbf{w} = e^{-x}\partial_u$. The first prolongation of this vector is given as follows

$$\mathbf{w}^{[1]} = \mathbf{w} + \phi^x \partial_{u_x} \tag{2.33}$$
$$= e^{-x}\partial_u - e^{-x}\partial_{u_x}. \tag{2.34}$$

The first-order differential invariant can be found by solving the characteristic equation:

$$\frac{dx}{0} = \frac{du}{e^{-x}} = -\frac{du_x}{e^{-x}}. \tag{2.35}$$

Therefore the set of first-order differential invariant is given by

$$y = x \quad , \quad w = u + u_x \tag{2.36}$$

and then, the set of third-order differential invariants is given by

$$y = x \quad , \quad w = u + u_x \quad , \quad w_y = u_x + u_{xx} \quad \text{and} \quad w_{yy} = u_{xx} + u_{xxx}. \tag{2.37}$$

We can now rewrite equation (2.32) in the form

$$w_{yy} + 2w_y + w = 0 \tag{2.38}$$

involving only the differential invariants. As expected the order is reduced by one.

One can readily see that the solution of equation (2.38) is given by

$$w = (c_0 + c_1 y)e^{-y}, \tag{2.39}$$

where c_0, c_1 are constants. So the right hand side expression in relation (2.36) becomes a first-order non homogeneous linear ordinary differential equation with constant coefficients:

$$u_x + u = (c_0 x + c_1)e^{-x}. \tag{2.40}$$

Using standard methods of integration it is readily seen that this equation has solution

$$u = \left(\frac{1}{2}c_0 x^2 + c_1 x + c_2\right) e^{-x}, \tag{2.41}$$

for some constant c_2, and this solution is therefore a group-invariant solution of equation (2.32).

Symmetries generate point transformations that leave the differential equation invariant. As a result, those point transformations can generate new solutions from existing solutions. In order to find these point transformations for a given generator $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$, we integrate the equations

$$\frac{\partial x^\star}{\partial \varepsilon} = \xi(x^\star, u^\star), \qquad \frac{\partial u^\star}{\partial \varepsilon} = \phi(x^\star, u^\star), \tag{2.42a}$$

subject to

$$x^\star|_{\varepsilon=0} = x, \qquad u^\star|_{\varepsilon=0} = u. \tag{2.42b}$$

2.4.6 Example. Using the previous example where $\mathbf{v} = e^{-x}\partial_u$ we get

$$\frac{\partial x^\star}{\partial \varepsilon} = 0, \qquad \frac{\partial u^\star}{\partial \varepsilon} = e^{-x^\star}, \tag{2.43}$$

which is equivalent to

$$x^\star = c_1, \tag{2.44}$$
$$u^\star = e^{-c_1}\varepsilon + c_2. \tag{2.45}$$

Conditions (2.42) give the values of c_1 and c_2 which allow us to write the point transformations as follow

$$x^\star = x \tag{2.46a}$$
$$u^\star = u + \varepsilon e^{-x}. \tag{2.46b}$$

2.5 Conclusion

We have illustrated the algorithm for finding symmetries, group-invariant solutions of an ODE and we have used one of the symmetries of an iterative equation of order 3 to obtain its exact solutions. This same symmetry was used to generate point transformations that leave the equation invariant.

3. Symmetry generator of third, fourth and nth-order differential equations in terms of the parameters r and s of the source equation

3.1 Introduction

In this chapter, the application of the algorithm for finding a symmetry of a linear iterative equation is considered. The aim of this chapter is to make a contribution to the results obtained by Krause and Michel [1], i.e. the expression of $\mathbf{v}$ in terms of the solutions u and v of the second-order source equation. Given that these solutions can be expressed in terms of the parameters r and s of the first-order source equation, we wish to see how these generators can be expressed in terms of the parameters r and s. Some properties of a linear iterative equation are used in order to generate the vector field that spanned the lie algebra in terms of the mentioned parameters for some special cases of r and A_2^2.

3.2 Symmetry Analysis of the linear iterative equation

3.2.1 Order 3. Consider the linear iterative equation of order three in reduced normal form

$$y^{(3)} + 4ay' + 2a'y = 0. \tag{3.1}$$

The third prolongation of $\mathbf{v} = \xi(x,y)\partial_x + \phi(x,y)\partial_y$ is given by

$$\mathbf{v}^{[3]} = \xi\partial_x + \phi\partial_y + \phi^x\partial_{y_x} + \phi^{xx}\partial_{y_{xx}} + \phi^{xxx}\partial_{y_{xxx}} \tag{3.2}$$

where ϕ^x, ϕ^{xx} and ϕ^{xxx} are given by (2.28). An application of the infinitesimal criterion of invariance gives

$$\mathbf{v}^{[3]}\left[y^{(3)} + 4ay' + 2a'y\right] = 0 \text{ whenever } \quad y^{(3)} + 4ay' + 2a'y = 0,$$

which reduces to

$$\xi\left(4a_xy' + 2a_{xx}y\right) + 2\phi a_x + 4\phi^x a + \phi^{xxx} = 0. \tag{3.3}$$

From equation (3.165) we get

$$y^{(3)} = -\left(4ay' + 2a'y\right). \tag{3.4}$$

Substituting ϕ^x and ϕ^{xxx} in (3.3) by their expression given by (2.28), and then using the expression of y_{xxx} in (3.4), equation (3.3) leads to the following single differential equation

$$\begin{aligned}
&\xi 4a_xy_x + \xi 2a_{xx}y + 2a_x\phi + 4a\phi_x + 4a\phi_yy_x - 4a\xi_xy_x - 4a\xi_y{y_x}^2 + \phi_{xxx} + (3\phi_{xxy} - \xi_{xxx})y_x \\
&+ (3\phi_{xyy} - 3\xi_{xxy}){y_x}^2 + (3\phi_{xy} - 3\xi_{xx})y_{xx} + (\phi_{yyy} - 3\xi_{xyy}){y_x}^3 + (3\phi_{yy} - 7\xi_{xy})y_xy_{xx} - \\
&(\phi_y - 3\xi_x)(4ay_x + 2a_xy) - \xi_{yyy}{y_x}^4 - 6\xi_{yy}{y_x}^2y_{xx} - 3\xi_y{y_{xx}}^2 - 4\xi_y(4ay_x + 2a_xy)y_x = 0.
\end{aligned}$$

Equating the coefficients of all powers of derivatives of y to zero yields the system of determining equations given as follows

$$1 : 2a_x\phi + 4a\phi_x + \phi_{xxx} + y\left[2\xi a_{xx} - 2a_x(\phi_y - 3\xi_x)\right] = 0 \tag{3.5a}$$

$$\begin{aligned} y_x &: 4\xi a_x + 4a\phi_y - 4a\xi_x - 8\xi_y a_x y \\ &\quad + (3\phi_{xxy} - \xi_{xxx}) - 4a(\phi_y - 3\xi_x) = 0 \end{aligned} \tag{3.5b}$$

$$y_{xx} : 3\phi_{xy} - 3\xi_{xx} = 0 \tag{3.5c}$$

$${y_x}^2 : -4a\xi_y + (3\phi_{xyy} - 3\xi_{xxy}) - 16\xi_y a = 0 \tag{3.5d}$$

$${y_x}^5 : \phi_{yyy} - 3\xi_{xyy} = 0 \tag{3.5e}$$

$$y_x y_{xx} : 3\phi_{yy} - 7\xi_{xy} = 0 \tag{3.5f}$$

$${y_x}^4 : -\xi_{yyy} = 0 \tag{3.5g}$$

$${y_x}^2 y_{xx} : -6\xi_{yy} = 0 \tag{3.5h}$$

$${y_{xx}}^2 : -3\xi_y = 0 \tag{3.5i}$$

Equations (3.5g), (3.5h) and (3.5i) show that

$$\xi = f(x), \tag{3.6}$$

where f is an arbitrary function.

Using (3.6) in (3.5f) we get

$$\phi = g(x)y + h(x), \tag{3.7}$$

for some functions g and h. From equation (3.5c) we have

$$g(x) = f'(x) + c_0 \tag{3.8}$$

for an arbitrary constant c_0. Using (3.5a), (3.5b), (3.7) and (3.8) the over determined system which allows to find ξ and ϕ is reduced to

$$\xi = f(x) \tag{3.9a}$$

$$\phi = (f'(x) + c_0)y + h(x) \tag{3.9b}$$

$$f''' + 4af' + 2a^{(1)}f = 0 \tag{3.9c}$$

$$f'''' + 4af'' + 6a^{(1)}f' + 2a^{(2)}f = 0 \tag{3.9d}$$

$$h^{(3)} + 4ah^{(1)} + 2a'h = 0. \tag{3.9e}$$

3.2.2 Remark. It should be noted that equation (3.9d) is the derivative of (3.9c).

Therefore, the infinitesimals are given by

$$\xi = f(x) \tag{3.10a}$$

$$\phi = (f'(x) + c_0)y + h(x), \tag{3.10b}$$

with f and g satisfying

$$f^{(3)} + 4af^{(1)} + 2a^{(1)}f = 0 \tag{3.10c}$$

$$h^{(3)} + 4ah^{(1)} + 2a'h = 0 \tag{3.10d}$$

respectively. So, f and h satisfy the original equation.

3.2.3 Order 4. Linear iterative equation of order four is

$$y^{(4)} + 10ay^{(2)} + 10a'y^{(1)} + (3a'' + 9a^2)y = 0. \tag{3.11}$$

The fourth prolongation of the infinitesimal generator is

$$\text{pr}^{(4)}\mathbf{v} = \xi\partial_x + \phi\partial_y + \phi^x\partial_{y_x} + \phi^{xx}\partial_{y_{xx}} + \phi^{xxx}\partial_{y_{xxx}} + \phi^{xxxx}\partial_{y_{xxxx}}, \tag{3.12}$$

where ϕ^{xxxx} is given by (2.28). Applying the fourth prolonged operator (3.12) to (3.11) yields

$$\phi^{xxxx} + 10a\phi^{xx} + 10a'\phi^x + (3a'' + 9a^2)\phi + \xi\left[10a'y_{xx} + 10a''y_x + (3a^{(3)} + 18aa')y\right] = 0.$$

Substituting the expressions of ϕ^{xxxx}, ϕ^{xx} and ϕ^x by their expressions, taking into consideration the substitution $y^{(4)} = -10ay^{(2)} - 10a'y^{(1)} - (3a'' + 9a^2)y$, we obtain

$$\begin{aligned}
&\phi_{xxxx} + (4\phi_{xxxy} - \xi_{xxxx})y_x + (6\phi_{xxyy} - 4\xi_{xxxy}){y_x}^2 + (4\phi_{xyyy} - 6\xi_{xxyy})y_x^3 + (\phi_{yyyy} - 4\xi_{yyyx})y_x^4 \\
&- \xi_{yyyy}y_x^5 + (6\phi_{xxy} - 4\xi_{xxx})y_{xx} + (12\phi_{xyy} - 18\xi_{xxy})y_xy_{xx} + (6\phi_{yyy} - 24\xi_{xyy})y_x^2y_{xx} + (3\phi_{yy} - \\
&12\xi_{xy})y_{xx}^2 + (4\phi xy - 6\xi_{xx})y_{xxx} + (4\phi_{yy} - 16\xi_{xy})y_xy_{xxx} - 10\xi_yy_{xx}y_{xxx} - 10\xi_{yy}y_x^2y_{xxx} + (\phi_y - \\
&4\xi_x)(-10ay_{xx} - 10a'y_x - (3a'' + 9a^2)y) - 5\xi_yy_x(-10ay_{xx} - 10a'y_x - (3a'' + 9a^2)y) - 10\xi_{yyy} \\
&y_x^3y_{xx} - 15\xi_{yy}y_xy_{xx}^2 + 10a\phi_{xx} + (2\phi_{xy} - \xi_{xx})10ay_x + 10a(\phi_{yy} - 2\xi_{xy})y_x^2 + 10a(\phi_y - 2\xi_x)y_{xx} \\
&- 10a\xi_{yy}y_x^3 - 30a\xi_yy_xy_{xx} + 10a'\phi_x + 10a'\phi_yy_x - 10a'\xi_xy_x - 10a'\xi_yy_x^2 + (3a'' + 9a^2)\phi \\
&+ 10a'\xi y_{xx} + 10a''\xi y_x + (3a^{(3)} + 18aa')\xi y = 0.
\end{aligned}$$

We are going to equate the coefficients of powers of derivatives of y to zero. It leads to the system below:

$$\begin{aligned}
1 :\ &\phi_{xxxx} + 10a\phi_{xx} + 10a'\phi_x + (3a'' + 9a^2)\phi \\
&+ y\left[(4\xi_x - \phi_y)(3a'' + 9a^2) + (3a^{(3)} + 18aa')\xi\right] = 0
\end{aligned} \tag{3.13a}$$

$$\begin{aligned}
y_x :\ &4\phi_{xxxy} - \xi_{xxxx} - 10a'(\phi_y - 4\xi_x) + 5(3a'' + 9a^2)\xi_yy \\
&+ 10a(2\phi_{xy} - \xi_{xx}) + 10a'\phi_y - 10a'\xi_x + 10a''\xi = 0
\end{aligned} \tag{3.13b}$$

$$y_x^2 : 6\phi_{xxyy} - 4\xi_{xxxy} + 50a'\xi_y + 10a(\phi_{yy} - 2\xi_{xy}) - 10a'\xi_y = 0 \tag{3.13c}$$

$$y_{xx} : 6\phi_{xxy} - 4\xi_{xxx} - 10a(\phi_y - 4\xi_x) + 10a(\phi_y - 2\xi_x) + 10a'\xi = 0 \tag{3.13d}$$

$${y_x}^3 : 4\phi_{xyyy} - 6\xi_{xxyy} - 10a\xi_{yy} = 0 \tag{3.13e}$$

$$y_xy_{xx} : 12\phi_{xyy} - 18\xi_{xxy} + 50a\xi_y - 310a\xi_y = 0 \tag{3.13f}$$

$$y_{xxx} : 4\phi_{yx} - 6\xi_{xx} = 0 \tag{3.13g}$$

$${y_x}^4 : \phi_{yyyy} - 4\xi_{xyyy} = 0 \tag{3.13h}$$

$${y_x}^2y_{xx} : 6\phi_{yyy} - 24\xi_{yyx} = 0 \tag{3.13i}$$

$${y_{xx}}^2 : 3\phi_{yy} - 12\xi_{xy} = 0 \tag{3.13j}$$

$$y_xy_{xxx} : 4\phi_{yy} - 16\xi_{xy} = 0 \tag{3.13k}$$

$$y_x^5 : -\xi_{yyyy} = 0 \tag{3.13l}$$

$$y_x^3y_{xx} : -10\xi_{yyy} = 0 \tag{3.13m}$$

$$y_x{y_{xx}}^2 : -15\xi_{yy} = 0 \tag{3.13n}$$

$$y_{xx}y_{xxx} : -10\xi_y = 0 \tag{3.13o}$$

$${y_x}^2y_{xxx} : -10\xi_{yy} = 0. \tag{3.13p}$$

Equations (3.13l) up to (3.13p) are equivalent to

$$\xi_y = 0, \tag{3.14}$$

that means

$$\xi = f(x) \tag{3.15}$$

for some function f. Using the latter in equations (3.13f), (3.145), (3.146), (3.13j) and (3.13k), we obtain

$$\phi_{yy} = 0 \tag{3.16}$$

which solution is

$$\phi = g(x)y + h(x) \tag{3.17}$$

for some functions g and h. Using equations (3.15) and (3.17) in equation (3.13g) we get

$$g(x) = \frac{3}{2}f'(x) + \alpha_0, \tag{3.18}$$

where α_0 is an arbitrary constant. Substituting (3.15),(3.17) and (3.18) into (3.13a), (3.13b) and (3.13d) we get

$$h^{(4)} + 10ah^{(2)} + 10a'h' + (3a'' + 9a^2)h = 0 \tag{3.19}$$

$$f^{(3)} + 4af' + 2a'f = 0, \tag{3.20}$$

$$f^{(4)} + 4a'f^{(2)} + 6a'f' + 2a^{(2)}f = 0 \tag{3.21}$$

$$f^{(5)} + 10af^{(3)} + 10a'f^{(2)} + 8(a'' + 3a^2)f' + (2a^{(3)} + 12aa')f = 0. \tag{3.22}$$

3.2.4 Remark. It can be verified that equation (3.21) is the derivative of (3.20), and (3.22) is the second derivative of (3.20) plus $6a$ times (3.20).

In all, the infinitesimals are given by

$$\xi = f(x) \tag{3.23}$$

$$\phi = \left(\frac{3}{2}f'(x) + \alpha_0\right)y + h(x), \tag{3.24}$$

where h satisfies equations (3.19), and f satisfies (3.20). So h satisfies the original equation but f still satisfies the iterative equation of order three.

3.2.5 For an arbitrary order n. Denoting as usual by $v = \xi(x,y)\partial_x + \phi(x,y)\partial_y$ the infinitesimal generator of the iterative equation of order n, [1] showed that its nth prolongation has the form

$$\text{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{k=0}^{n} \phi^{[k]}(x, y^{(n)})\partial_{y^{(k)}}, \tag{3.25}$$

where

$$\begin{aligned}\phi^{[n]} =&(\phi_y - k\xi_x)y^{(k)} - (k+1)\xi_y y' y^{(k)} - \binom{k+1}{2}\xi_y y'' y^{(k-1)} + k(\phi_{yy} - k\xi_{xy})y'y^{(k-1)} \\ &+ k(\phi_{xy} - \frac{k-1}{2}\xi_{xx})y^{(k-1)} + \frac{k}{2}\left(\phi_{xxy} - \frac{k-2}{3}\xi xxx\right) y^{(k-2)} + \dots .\end{aligned} \tag{3.26}$$

Using the invariance criterion and separating by the powers of derivatives of y they proved that the most general form of the symmetry generator is given by

$$\mathbf{v} =f(x)\partial_x + \left[\left(\frac{n-1}{2}f'(x) + c\right) y + h(x)\right]\partial_y, \tag{3.27}$$

where

$$\binom{n+1}{3} f''' + 4A_n^2 f' + 2A_n^{2\prime} f = 0 \tag{3.28a}$$

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0 \tag{3.28b}$$

and c is an arbitrary constant. To obtain the same condition on h, [2] used the Leibniz's rule of differentiating a product to rewrite (3.26) fully. The showed that (3.26) is the same as

$$\phi^{[j]} = \left(\left[\frac{n-1}{2}f' + \alpha\right] y\right)^{(j)} + h^{(j)} - \sum_{i=1}^{j}\binom{j}{i} y^{j+1-i}\xi^i, \quad j = 1, \dots n \tag{3.29}$$

Using the above result and the invariance criterion they showed that h satisfies the original equation.

However, using in

$$\binom{n+1}{3} f''' + 4A_n^2 f' + 2A_n^{2\prime} f = 0 \tag{3.30}$$

the expression of A_n^2 given in (1.10), i.e.

$$A_n^2 = \binom{n+1}{3} A_2^2, \tag{3.31}$$

we get

$$\binom{n+1}{3} f''' + 4\binom{n+1}{3} A_2^2 f' + 2\binom{n+1}{3} A_2^{2\prime} f = 0, \tag{3.32}$$

which can aslo be written as

$$f''' + 4A_2^2 f' + 2A_2^{2\prime} f = 0. \tag{3.33}$$

Hence, f satisfies the third order linear iterative equation as in the cases of order 3 and 4. Therefore, condition on f does not depend on the order of the linear iterative equation. In all, the most general form of the symmetry generator is given by

$$\mathbf{v} =f(x)\partial_x + \left[\left(\frac{n-1}{2}f'(x) + c\right) y + h(x)\right]\partial_y, \tag{3.34}$$

where the new condition on f is

$$f''' + 4A_2^2 f' + 2A_2^{2\prime} f = 0 \tag{3.35a}$$

and

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} y^{(i)} = 0. \tag{3.35b}$$

Equations (3.35) are linear iterative equations with the same source equation $y'' + A_2^2 y = 0$.

Based on the properties of iterative equations already outlined, finding the solutions of (3.35) will be reduced to finding the solutions of the second-order source equation. It is well-known that if we assume that u and v are solutions of the second-order source equation, then n linearly independent solutions of (1.13) are given by [1]

$$y_k = u^{n-(k+1)} v^k \qquad 0 \leq k \leq n-1. \tag{3.36}$$

Let us verify this known fact for linear iterative equations of order $n = 3, 4, 5, 6$.

- For $n = 3$. Suppose indeed that u and v are the two linearly independent solutions of

$$y'' + ay = 0, \tag{3.37}$$

and let us check that $y_k = u^{2-k} v^k, 0 \leq k \leq 2$, are linearly independent solutions of the third-order linear iterative equation

$$y^{(3)} + 4ay' + 2a'y = 0. \tag{3.38}$$

We have

$$y_k^{(3)} = (u^{2-k})^{(3)} v^k + 3(u^{2-k})^{(2)} v^{k\prime} + 3(u^{2-k})' v^{k(2)} + u^{2-k}(v^k)^{(3)} \tag{3.39}$$

$$\begin{aligned}
&= [(2-k)u^{(3)}u^{1-k}v^k + 3(2-k)(1-k)u'u^{(2)}u^{-k}v^k \\
&\quad + (2-k)(1-k)(-k)u'^3 u^{-k-1}v^k] + 3[k(2-k)u^{(2)}u^{1-k}v'v^{k-1} \\
&\quad + k(2-k)(1-k)u'^2 u^{-k} v' v^{k-1}] + 3[k(2-k)u'u^{1-k}v^{(2)}v^{k-1} \\
&\quad + k(2-k)(k-1)u'u^{1-k}v'^2 v^{k-2}] + [ku^{2-k}v^{(3)}v^{k-1} \\
&\quad + 3k(k-1)u^{2-k}v'v^{(2)}v^{k-2} + (k-2)(k-1)ku^{2-k}v'^3 v^{k-3}]
\end{aligned} \tag{3.40}$$

Using in (3.40) the substitutions $f'' = -af$, $f''' = -(af)'$ for $f = u, v$ gives

$$\begin{aligned}
(y_k)^{(3)} = &- a'(2-k)u^{2-k}v^k - a(2-k)u'u^{1-k}v^k \\
&- 3a(2-k)(1-k)u'u^{1-k}v^k - k(1-k)(2-k)u'^3 u^{-1-k} v^k \\
&- 3ak(2-k)v'u^{2-k}v^{k-1} + 3k(2-k)(1-k)u'^2 u^{-k} v' v^{k-1} \\
&- 3ak(2-k)u'u^{1-k}v^k + 3k(k-1)(2-k)u'u^{1-k}v'^2 v^{k-2} \\
&- a'ku^{2-k}v^k - akv'u^{2-k}v^{k-1} - 3ak(k-1)u^{2-k}v'v^{k-1} \\
&+ (k-2)(k-1)(k)u^{2-k}v'^3 v^{k-3}
\end{aligned} \tag{3.41}$$

Substituting (3.41) into (3.38) and expressing also y_k' and y_k in the resulting equation in terms of u and v gives

$$\begin{aligned}(y_k)^{(3)} + 4a(y_k)' + 2a'(y_k) =& k(k-1)(k-2)(-u'^3u^{-1-k}v^k + 3u'^2u^{-k}v'v^{k-1} \\ &+ 3u'u^{1-k}v'^2 + v^{2-k} + u^{2-k}v'^3v^{k-3}) \\ =& k(k-1)(k-2)u^{-1-k}v^{k-3}[(uv' - u'v)^3] \\ =& \left(\prod_{j=0}^{2}(k-j)\right) \cdot u^{5-k}v^{k-3}\left[\left(\frac{v}{u}\right)'\right]^3 \\ =& 0, \qquad \textit{for} \qquad k = 0, 1, 2.\end{aligned}$$

On the other hand, the Wronskian of the three functions y_k is $-2(vu' - v'u)^3 \neq 0$, showing that the functions y_k, $0 \leq k \leq 2$, are linearly independent.

Let Ω_n be the linear operator corresponding to the linear iterative equation of order n with source equation $y'' + ay = 0$. Thus $\Omega_3 = \dfrac{d^3}{dx^3} + 4a\dfrac{d}{dx} + 2a'$. Let y_k, for $0 \leq k \leq n-1$ be given as above by $y_k = u^{n-(k+1)}v^k$. We now want to express $\Omega_4, \Omega_5, \Omega_6$ in a similar way, and then generate a general expression for Ω_n.

- For $n = 4$ The iterative equation of order 4 is given by

$$y^{(4)} + 10ay^{(2)} + 10a'y^{(1)} + (3a'' + 9a^2)y = 0 \tag{3.42}$$

Here, we have $\Omega_4 = \dfrac{d^4}{dx^4} + 10a\dfrac{d^2}{dx^2} + 10a'\dfrac{d}{dx} + (3a'' + 9a^2)$ and let y_k, for $0 \leq k \leq 3$ is given by $y_k = u^{3-k}v^k$. Hence,

$$\begin{aligned}(y_k)^{(4)} =& \left(u^{3-k}\right)^{(4)}v^k + 4\left(u^{3-k}\right)^{(3)}(v^k)^{(1)} + 6\left(u^{3-k}\right)^{(2)}(v^k)^{(2)} \\ &+ 4\left(u^{3-k}\right)^{(1)}(v^k)^{(3)} + u^{3-k}(v^k)^{(4)} \\ =& (3-k)[u^{(4)}u^{2-k} + 4(2-k)u'u^{1-k}u^{(3)} + 3(2-k){u^{(2)}}^2u^{1-k} \\ &+ 6(2-k)(1-k)u^{(2)}u'^2u^{-k} - k(2-k)(1-k)u'^4u^{-k-1}]v^k \\ &+ 4(3-k)[u^{(3)}u^{2-k} + 3(2-k)u'u^{(2)}u^{1-k} + (2-k)(1-k) \\ &\times u'^3u^{-k}][kv'v^{k-1}] + 6(3-k)[u^{(2)}u^{2-k} + (2-k)u'^2u^{1-k}] \\ &\times k[v^{(2)}v^{k-1} + (k-1)v'^2v^{k-2}] + 4(3-k)[u'u^{2-k}] \\ &\times k[v^{(3)}v^{k-1} + 3(k-1)v'v^{(2)}v^{k-2} + (k-1)(k-2)v'^3v^{k-3}] \\ &+ u^{3-k}k\left[v^{(4)}v^{k-1} + 4(k-1)v'v^{(3)}v^{k-2} + 3(k-1){v^{(2)}}^2v^{k-2}\right. \\ &\left.+ 6(k-1)(k-2)v^{(2)}v'^2v^{k-3} + (k-1)(k-2)(k-3)v'^4v^{k-4}\right].\end{aligned} \tag{3.43}$$

Similarly,

$$\begin{aligned}(y_k)^{(2)} =& (3-k)u^{(2)}u^{2-k}v^k + (3-k)(2-k)u'^2u^{1-k}v^k + k(3-k)u'u^{2-k} \\ &\times v'v^{k-1} + ku^{3-k}v^{(2)}v^{k-1} + k(k-1)u^{3-k}v'^2v^{k-2} \\ &+ k(3-k)u'u^{2-k}v'v^{k-1}\end{aligned} \tag{3.44}$$

and

$$(y_k)^{(1)} = (3-k)u'u^{2-k}v^k + kv'v^{k-1}u^{3-k}. \tag{3.45}$$

One can readily see that if a function f satisfies $y'' + a\,y = 0$ then,

$$f''' = -a'f - af' \tag{3.46a}$$
$$f'''' = -2a'f' + (a^2 - a'')f. \tag{3.46b}$$

Using in (3.43), (3.44), (3.45) the substitutions (3.46) for $f = u, v$ and plugging the resulting expression in (3.42) lead to

$$\begin{aligned}
\Omega_4(y_k) = {} & (3-k)[-2a'u' + (a^2 - a''u)]u^{2-k}v^k + 4(3-k)(2-k)u'u^{1-k}(-a'u - au')v^k + \\
& 3(3-k)(2-k) - au^2u^{1-k}v^k + 6(3-k)(2-k)(1-k)[-au]u'^2u^{-k}v^k - k(3-k) \\
& (2-k)(1-k)u'^4u^{-k-1}v^k + 4k(3-k)v'v^{k-1}[(-a'u - au')u^{2-k} + 3(2-k)u' \times \\
& (-au)u^{1-k} + (2-k)(1-k)u'^3u^{-k}] + 6k(3-k)[(-au)u^{2-k} + (2-k)u'^2u^{1-k}] \\
& \times k[(-av)v^{k-1} + (k-1)v'^2v^{k-2}] + 4k(3-k)u'u^{2-k}[(-a'v - av')v^{k-1} - \\
& 3(k-1)v'(avv^{k-2}) + (k-1)(k-2)v'^3v^{k-3}] + ku^{3-k}[((a^2 - a'')v - 2a'v')v^{k-1} + \\
& 4(k-1)v'v^{k-2}(-a'v - av') + 3(k-1)(a^2v^2)v^{k-2} - 6(k-1)(k-2)(av)v'^2v^{k-3} \\
& + (k-1)(k-2)(k-3)v'^4v^{k-4}] - 10(3-k)(a^2u)u^{2-k}v^k + 10a(3-k)(2-k)u'^2 \\
& u^{1-k}v^k + 10ak(3-k)u'u^{2-k}v'v^{k-1} - 10ku^{3-k}(a^2v)v^{k-1} + 10ak(k-1)v'^2v^{k-2}u^{3-k} \\
& + 10ak(3-k)u'u^{2-k}v'v^{k-1} + 10a'(3-k)u'u^{2-k}v^k + 10a'ku^{3-k}v'v^{k-1} \\
& + (3a'' + 9a^2)u^{3-k}v^k.
\end{aligned}$$

$$\begin{aligned}
= & k(k-1)(k-2)(k-3)[u^{3-k}v'^4v^{k-4} - 4u'u^{2-k}v'^3v^{k-3} \\
& + 6u'^2v'^2u^{1-k}v^{k-2} - 4u'^3u^{-k}v'v^{k-1} - u'^4u^{-k-1}v^k]
\end{aligned} \tag{3.47}$$

$$= \left(\prod_{j=0}^{3}(k-j)\right) \cdot u^{7-k}v^{k-4}\left[\left(\frac{v}{u}\right)'\right]^4$$

$$=0, \quad \text{for} \quad k = 0, 1, 2, 3.$$

Therefore,

$$\Omega_4(y_k) = \left(\prod_{j=0}^{3}(k-j)\right) \cdot u^{7-k}v^{k-4}\left[\left(\frac{v}{u}\right)'\right]^4, \quad 0 \le k \le 3. \tag{3.48}$$

Proceeding in the same way for $n = 5$ and $n = 6$, and for the corresponding values of y_k shows that

- $$\Omega_5(y_k) = \left(\prod_{j=0}^{4}(k-j)\right) \cdot u^{9-k} v^{k-5} \left[\left(\frac{v}{u}\right)'\right]^5, \quad 0 \leq k \leq 4 \tag{3.49}$$

  and

- $$\Omega_6(y_k) = \left(\prod_{j=0}^{5}(k-j)\right) \cdot u^{11-k} v^{k-6} \left[\left(\frac{v}{u}\right)'\right]^6, \quad \text{for} \quad 0 \leq k \leq 5 \tag{3.50}$$

which are equal to zero for $0 \leq k \leq 5$. It clearly follows from the expressions of $\Omega_n(y_k)$ obtained for $n = 3, 4, 5, 6$ that the general expression for an arbitrary $n \geq 3$ is

$$\Omega_n(y_k) = \left(\prod_{j=0}^{n-1}(k-j)\right) \cdot u^{2n-1-k} v^{k-n} \left[\left(\frac{v}{u}\right)'\right]^n = 0, \quad \text{for} \quad 0 \leq k \leq n-1, \quad n \geq 3. \tag{3.51}$$

A formal proof of the validity of (3.51) could be done by induction on n.

We deduce from (3.36) that the solutions of (3.35) are given by

$$f(x) = c_1 u^2 + c_2 uv + c_3 v^2 \tag{3.52a}$$

$$h(x) = \sum_{k=4}^{n+3} c_k u^{n-1-k} v^k, \tag{3.52b}$$

where u and v are solutions of (1.14).

Therefore, the general infinitesimal symmetry generator $\mathbf{v} = \xi\partial_x + \phi\partial_y$ of the linear iterative equation of order n is given by

$$\xi(x) = c_1 u^2 + c_2 uv + c_3 v^2 \tag{3.53a}$$

$$\phi(x,y) = \left[\frac{n-1}{2}\left(2c_1 u'u + c_2 u'v + c_2 uv' + 2c_3 vv'\right) + c_0\right] y + \sum_{k=4}^{n+3} c_k u^{n-1-k} v^k \tag{3.53b}$$

where $c_0, \ldots, c_{n+3}$ are arbitrary constants. There are $n+4$ arbitrary constants, meaning that the Lie algebra has maximal dimension. Letting v_k be the generators obtained by setting $c_j = \delta_j^k$ in (3.53) allows us to find the $n+4$ vector fields [1] (although this result is not an original one of [1])

$$v_0 = y\partial_y \tag{3.54a}$$

$$v_1 = u^2\partial_x + (n-1)uu'y\partial_y \tag{3.54b}$$

$$v_2 = uv\partial_x + \frac{n-1}{2}(u'v + uv')y\partial_y \tag{3.54c}$$

$$v_3 = v^2\partial_x + (n-1)vv'y\partial_y \tag{3.54d}$$

$$v_k = u^{n-1-k} v^k \partial_y \quad , k = 4, \ldots n+3 \tag{3.54e}$$

that span the Lie algebra. This has been obtained in [1] by a slightly different method. Note indeed that this is simply based on the substitution of (3.52) into (3.34), which was clearly obtained in [1]. We can notice that finding symmetries and solutions of linear iterative equations reduce to the case of the second-order source equation in normal form $y'' + A_2^2 y = 0$. Despite the low order of the source equation, its solutions u and v are generally not available and in this section, we are interested in finding explicit solutions of the equation in terms of the parameter r of the source equation, and for r as large (i.e. arbitrary) as possible. Recall that

$$A_2^2 = \frac{r'^2 - 2rr''}{4r^2} \quad \text{provided that} \quad s = -\frac{n-1}{2} r'. \tag{3.55}$$

3.3 Solutions of the source equation for given values of r

We have noted that the solutions of the source equation $y'' + ay = 0$ are not available for A_2^2 arbitrary. So we are interested in the same problem from a different angle. For given values of the parameter r of the source equation, find the solutions u and v in terms of r. In view of (3.54) this amounts to solving

$$\frac{r'^2 - 2rr''}{4r^2} = A_2^2, \tag{3.56}$$

for r. So this problem is not easier than that of finding u and v directly, but here one is only interested in equations determined by the source parameter r, and to finding u and v in terms of r. Note also that in vertue of (3.54) it is not necessary to find an expression of the symmetries in terms of r (it only suffices to express u and v in terms of r).

3.3.1 Case $r =$ constant.

Values of s, A_2^2, u and v : Invoking equations (3.56) we have

$$s = 0, \qquad A_2^2 = 0, \tag{3.57}$$

so that the corresponding source equation is $y'' = 0$. And then, the two linearly independent solutions are

$$u = 1, \qquad v = x. \tag{3.58}$$

These solutions can not be expressed in terms of the parameters r and s. However, we can write the symmetries as functions of the variables x and y, in this somewhat trivial case.

Symmetry generators : We use (3.54) and (3.58) to generate the vectors that spanned the lie algebra:

$$v_0 = y\partial_y \tag{3.59a}$$
$$v_1 = \partial_x \tag{3.59b}$$
$$v_2 = x\partial_x + \frac{n-1}{2} y\partial_y \tag{3.59c}$$
$$v_3 = x^2\partial_x + (n-1)x\partial_y \tag{3.59d}$$
$$v_k = x^k\partial_y \quad , k = 4, \dots n+3. \tag{3.59e}$$

3.3.2 Case $r(x) = c_1x + c_2$, $c_1 \neq 0$.

Values of s, A_2^2, u and v: For such value of r , we have

$$s = -\frac{n-1}{2}c_1, \tag{3.60}$$

and

$$A_2^2(x) = \frac{(c_1x+c_2)'^2 - 2\,(c_1x+c_2)\,(c_1x+c_2)''^2}{4(c_1x+c_2)^2} \tag{3.61}$$

Then now,

$$A_2^2(x) = \left(\frac{c_1}{2c_1x+2c_2}\right)^2. \tag{3.62}$$

Equation (1.14) becomes

$$y'' + \left(\frac{c_1}{2c_1x+2c_2}\right)^2 y = 0 \tag{3.63}$$

and this equation is of Euler's type. Let $c_1x + c_2 = e^{2t}$, then

$$dx = e^t dt, \tag{3.64a}$$

$$y'(x) = \frac{c_1}{2}e^{-2t}y'(t) \tag{3.64b}$$

$$y''(x) = -\frac{c_1^2}{2}e^{-4t}y'(t) + \frac{c_1^2}{4}e^{-4t}y''(t). \tag{3.64c}$$

Substituting back (3.64) in (3.87) we get

$$y'' - 2y' + y = 0, \tag{3.65}$$

and its characteristic equation is

$$Q^2 - 2Q + 1 = 0. \tag{3.66}$$

Thus, the solutions of (3.65) are

$$\exp(t), \qquad \text{and} \qquad t\,\exp(t). \tag{3.67}$$

Substituting back $t = \frac{1}{2}\ln(c_1x + c_2) = \ln\sqrt{r}$ in the above solutions, we get the two linearly independent solutions of equation (3.87):

$$u = \sqrt{r}, \qquad v = \sqrt{r}\ln r. \tag{3.68}$$

Symmetry generators in terms of the parameters r and s: Similarly, invoking (3.54) , the $n+4$ vectors that spanned the Lie algebra in this special case are given by

$$v_0 = y\partial_y \tag{3.69a}$$

$$v_1 = r\partial_x - sy\partial_y \tag{3.69b}$$

$$v_2 = r\ln r\partial_x - (1+\ln r)sy\partial_y \tag{3.69c}$$

$$v_3 = r\ln^2 r\partial_x - \left(\ln^2 r + 2\ln r\right) sy\partial_y \tag{3.69d}$$

$$v_k = r^{\frac{n-1}{2}}\ln^k r\partial_y \quad , k = 4, \ldots n+3. \tag{3.69e}$$

3.3.3 Case $r(x) = (c_1x + c_2)^2$, $c_1 \neq 0$.

Corresponding values of s, A_2^2, u and v: Here, we have

$$r = (c_1x + c_2)^2, \qquad s = -c_1(n-1)\sqrt{r}, \qquad A_2^2 = 0, \tag{3.70}$$

and equation (1.14) becomes

$$y'' = 0. \tag{3.71}$$

Let

$$u = 1 \qquad\qquad v = c_1x + c_2 \tag{3.72}$$

be the two linearly independent solutions. From equation (3.70), we have

$$c_1x + c_2 = \sqrt{r}. \tag{3.73}$$

Therefore, the two linearly independent solutions u and v (given in terms of r) are as follow

$$u = 1 \qquad\qquad v = \sqrt{r}. \tag{3.74}$$

Symmetry generators in terms of the parameters r and s: According to (3.54), the expressions of the symmetries in terms of the parameter r and s are then given by

$$v_0 = y\partial_y \tag{3.75a}$$
$$v_1 = \partial_x \tag{3.75b}$$
$$v_2 = \sqrt{r}\partial_x - \frac{s}{2\sqrt{r}}y\partial_y \tag{3.75c}$$
$$v_3 = r\partial_x - sy\partial_y \tag{3.75d}$$
$$v_k = r^{\frac{k}{2}}\partial_y \quad , k = 4, \ldots n+3. \tag{3.75e}$$

3.3.4 Case $r(x) = (c_1x + c_2)^m$, $c_1 \neq 0, m \neq 0, 1, 2$.

Corresponding values of s, A_2^2 and the linearly independent solutions: To find the value of A_2^2 generated by

$$r(x) = (c_1x + 2c_2)^m, \tag{3.76}$$

we just need to substitute its expression in the expression

$$A_2^2(x) \equiv \frac{r'^2 - 2rr''}{4r^2}. \tag{3.77}$$

The corresponding value of A_2^2 is thus given by

$$A_2^2(x) = \left(2m - m^2\right)\left(\frac{c_1}{2c_1x + 2c_2}\right)^2, \tag{3.78}$$

and equation (1.14) becomes

$$y(x)'' + \left(2m - m^2\right)\left(\frac{c_1}{2c_1 x + 2c_2}\right)^2 y = 0. \tag{3.79}$$

Using the change of variable $c_1 x + c_2 = \exp \frac{2t}{\sqrt{2m-m^2}}$ we get $dx = \frac{2}{2\sqrt{2m-m^2}} \exp \frac{2t}{\sqrt{2m-m^2}} dt$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \left[\frac{c_1\sqrt{2m-m^2}}{2\exp \frac{2t}{\sqrt{2m-m^2}}}\right]\frac{dy}{dt} \tag{3.80a}$$

and

$$\frac{d^2y}{dx^2} = \frac{dy'}{dt}\frac{dt}{dx} = \left[\frac{c_1^2\left(2m-m^2\right)}{4\exp\left(\frac{4t}{\sqrt{2m-m^2}}\right)}\right]y'' + \left[-\frac{c_1^2\sqrt{2m-m^2}}{2\exp\left(\frac{4t}{\sqrt{2m-m^2}}\right)}\right]y'(t). \tag{3.80b}$$

Using (3.80) into (3.79) leads to

$$\frac{d^2y}{dt^2} - \frac{2}{\sqrt{2m-m^2}}\frac{dy}{dt} + y = 0 \tag{3.81}$$

and the solutions of the corresponding characteristic equation are given by

$$\lambda_1 = \sqrt{\frac{2-m}{m}}, \quad \lambda_2 = -\sqrt{\frac{m}{2-m}}, \tag{3.82}$$

we assume that $2m - m^2 > 0$ in order to avoid complex numbers. Therefore, the solutions of (3.79), in terms or the parameter r, are given as follow

$$u = r^{\frac{1}{2}}, \quad v = r^{\frac{2-m}{2m}} \tag{3.83}$$

Symmetry generators in terms of the parameters r and s: We can now express the symmetries in terms of the parameters r and s since we were able to express the two solutions in terms of these parameters. Hence, the $n+4$ symmetries are as follow

$$v_0 = y\partial_y \tag{3.84a}$$
$$v_1 = r\partial_x - sy\partial_y \tag{3.84b}$$
$$v_2 = r^{\frac{1}{m}}\partial_x - \frac{2}{m(n-1)}r^{\frac{1-m}{m}}sy\partial_y \tag{3.84c}$$
$$v_3 = r^{\frac{2-m}{m}}\partial_x - \left(\frac{2-m}{m}\right)r^{\frac{2-2m}{m}}sy\partial_y \tag{3.84d}$$
$$v_k = r^{\frac{k}{m}+\frac{n-1-2k}{2}}\partial_y \quad, k = 4, \dots n+3. \tag{3.84e}$$

3.3.5 Case $r(x) = \exp(mx), m \neq 0$.

Corresponding value of A_2^2 and the linearly independent solutions: If the parameter r is an exponential function of the above form then

$$s = -\frac{(n-1)m}{2} r. \tag{3.85}$$

Equation (1.16) enables us to find the expression of A_2^2. Hence,

$$A_2^2(x) = -\frac{m^2}{4}. \tag{3.86}$$

Equation (1.14) becomes

$$y'' - \left(\frac{m}{2}\right)^2 y = 0. \tag{3.87}$$

and has

$$u = r^{\frac{1}{2}}, \qquad v = r^{-\frac{1}{2}} \tag{3.88}$$

as solutions.

Symmetries generator in terms of the parameters r and s: The knowledge of u, v allows us to find the symmetries as follows

$$v_0 = y\partial_y \tag{3.89a}$$
$$v_1 = r\partial_x - sy\partial_y \tag{3.89b}$$
$$v_2 = \partial_x \tag{3.89c}$$
$$v_3 = \frac{1}{r}\partial_x + \left(\frac{s}{r^2}\right) y\partial_y \tag{3.89d}$$
$$v_k = r^{\frac{n-1-2k}{2}}\partial_y \quad, k = 4, \dots n+3. \tag{3.89e}$$

3.3.6 Case $r(x) = \ln x$. In this case, the expression for A_2^2 is given by

$$A_2^2 = \frac{1 + 2\ln x}{4x^2 \ln^2 x}. \tag{3.90}$$

Solving

$$y'' + \left(\frac{1 + 2\ln x}{4x^2 \ln^2 x}\right) y = 0 \tag{3.91}$$

is not easy and we have tried to find the solutions using *Mathematica* (a popular computing system) which gives us

$$u = G_{1,2}^{2,0}\left(\frac{r}{2} \quad \begin{matrix} \frac{3}{2} \\ \frac{1}{2}, \frac{1}{2} \end{matrix}\right), \qquad v = \frac{\sqrt{r}}{\sqrt{2}}, \tag{3.92}$$

where $G_{p,q}^{m,n}$ is the Meijer G-function define by

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \, ds, \tag{3.93}$$

3.4 Solutions of the source equation for given values of A_2^2

As already noted, the solutions of a general linear iterative equation are easily obtained using the simple formula (3.36) from the linearly independent solutions u and v of the second-order source equation. In the previous section, we obtained expressions of u and v in terms of r, for various large values of r. In this present section, to find u and v, instead of choosing a value for r, we rather specify a value for A_2^2. Our aim is to solve the source equation for A_2^2 as large as possible. Despite its simplicity, the general solutions of the second-order source equation $y'' + A_2^2 y = 0$ is not known, for A_2^2 arbitrary, and to solve this equation for specific values of A_2^2 we shall make use of the following theorem proved in [15] about the method for solving $y^{(k)} - f(x)y = 0$. Only values of A_2^2 that have not been obtained in the previous section by a choice of r will be considered.

3.4.2 Theorem. *Suppose f is continuous on $[a, b], c \in [a, b]$, and k is a natural number. Define the sequence of functions $\{f_n(x)\}_{n=0}^{\infty}$ by*

$$f_o(x) = a_0^{(0)} + a_1^{(0)} x + \cdots + a_{k-1}^{(0)} x^{k-1} \neq 0, \tag{3.94}$$

$$\begin{aligned} f_n(x) = &\int_c^x \int_c^{u_{k-1}} \cdots \int_c^{u_1} f_{n-1}(u) f(u) du du_1 \dots du_{k-1} \\ &+ \sum_{j=0}^{k-1} a_j^{(n)} x^j, \quad n = 1, 2, \dots. \end{aligned} \tag{3.95}$$

where $a_0^{(n)}, \dots, a_{k-1}^{(n)}$, are constants, $n = 0, 1, 2, \dots$ If the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ to some function $S(x)$ then $\sum_{n=0}^{\infty} f_n^{(j)}(x)$ converges uniformly to $S^{(j)}(x)$ for $a \leq x \leq b$, $j = 1, 2, \dots k$ and $S^{(k)}(x) = S(x) f(x)$ on [a,b].

The text of the above theorem in [15] shows that such a sequence of functions satisfy

$$f_n''(x) = f_{n-1}(x) f(x) \tag{3.96}$$

for $k = 2$.

3.4.3 Case $A_2^2 = 0$. To avoid redoing the same things let us find the values of r such that $A_2^2 = 0$. To find the values of the parameter r that generate $A_2^2 = 0$ we just need to solve the equation

$$A_2^2(x) \equiv \frac{r'^2 - 2rr''}{4r^2} = 0. \tag{3.97}$$

The solutions of (3.97) are given by

$$r(x) = (c_1 x + 2c_2)^2, \tag{3.98}$$

where c_1 and c_2 are arbitrary constants. Indeed, for such value of r the coefficient $A_2^2 = 0$. This case has already been treated in sections 3.3.1 and 3.3.3 .

3.4.4 Case $A_2^2 = C$, for an arbitrary constant $C \neq 0$. Let

$$y'' + Cy = 0 \tag{3.99}$$

be the second-order source equation (in normal form) generating a family of iterative equation. First we note that since C is a constant, and since each equation in the corresponding family of iterative equations (in normal form) has coefficients which are only polynomials in C and its derivatives, it follows that each iterated equation has constant coefficients. The converse is also clear from the relation

$$A_n^2 = \binom{n+1}{3} A_2^2 = \frac{n(n-1)(n+1)}{6} A_2^2 = \frac{n^3 - n}{6} A_2^2. \tag{3.100}$$

Consequently each iterated equation has constant coefficient if and only if A_2^2 is a constant. In [13] they proved that indeed, all coefficients in each iterated equation depends solely on A_n^2 (thus on A_2^2) and its derivatives. Now, let

$$\Delta_n(y) = 0 \tag{3.101}$$

be the iterative equation of order n generated by (3.99), and suppose that its labelling coefficient A_n^2 is a constant, B say. To find the parameter r of the source equation that generates (3.101), one needs to solve the equation

$$B = \frac{n^3 - n}{6} C, \quad \text{i.e.} \quad B = \frac{n^3 - n}{6} \left(\frac{r'^2 - 2rr''}{4r^2} \right) \tag{3.102}$$

for r. This has solution

$$r(x) = c_2 \cos^2 \left[\sqrt{6} \sqrt{\frac{B}{-n + n^3}} (-2c_1 + x) \right], \tag{3.103}$$

and the corresponding value of $s(x) = -(1/2)(n-1)r'(x)$ has expression

$$s(x) = \sqrt{\frac{3}{2}} c_2 (-1 + n) \sqrt{\frac{B}{-n + n^3}} \sin \left[2\sqrt{6} \sqrt{\frac{B}{-n + n^3}} (-2c_1 + x) \right]; \tag{3.104}$$

However, using the usual operator Φ_n and r alone, one can easily generate iterative equations of any order n with $A_n^2 = B$. For instance the list of canonical normal forms of iterative equations of order 3 to 10 with constant coefficient is given by

$$
\begin{aligned}
& y^{(2)} + A_2^2 y = 0, \\
& y^{(3)} + 4A_2^2 y' = 0, \\
& y^{(4)} + 10A_2^2 y'' + 9\left(A_2^2\right)^2 y = 0, \\
& y^{(5)} + 20A_2^2 y^{(3)} + 64\left(A_2^2\right)^2 y' = 0, \\
& y^{(6)} + 35A_2^2 y^{(4)} + 259\left(A_2^2\right)^2 y'' + 225\left(A_2^2\right)^3 y = 0, \\
& y^{(7)} + 56A_2^2 y^{(5)} + 784\left(A_2^2\right)^2 y^{(3)} 2304\left(A_2^2\right)^3 y' = 0, \\
& y^{(8)} + +84A_2^2 y^{(6)} + 1974\left(A_2^2\right)^2 y^{(4)} + 12916\left(A_2^2\right)^3 y'' + 11025\left(A_2^2\right)^4 y = 0, \\
& y^{(9)} + 120A_2^2 y^{(7)} + 4368\left(A_2^2\right)^2 y^{(5)} + +52480\left(A_2^2\right)^3 y^{(3)} 147456\left(A_2^2\right)^4 y' = 0, \\
& y^{(10)} + 165A_2^2 y^{(8)} + 8778\left(A_2^2\right)^2 y^{(6)} + 172810\left(A_2^2\right)^3 y^{(4)} + 1057221\left(A_2^2\right)^4 y'' \\
& + 893025\left(A_2^2\right)^5 y = 0.
\end{aligned}
\tag{3.105}
$$

Using the standard change of variable

$$
y = w \exp\left(-\frac{1}{n}\int_{z_0}^{z} B_{n-1} dv\right), \tag{3.106}
$$

we can transform these equations into their standard forms, in which they will thus depend on two (and not just one) arbitrary coefficient, B_{n-1} and $B = B_{n-2}$. For instance, for $n = 3$, after transformation and normalization of the coefficient of y', the equation has the form

$$
\frac{1}{27}(9B_1B_2 - 2{B_2}^3)w + B_1 w' + B_2 w'' + w^{(3)} = 0. \tag{3.107}
$$

For expressing the solutions of (3.101) in terms of r, one may proceed as follows. If we let

$$
u = \cos[\sqrt{C}x], \quad \text{and} \quad v = \sin[\sqrt{C}x] \tag{3.108}
$$

be the linear iterative solutions to $y'' + Cy = 0$. For instance, in terms of r, we have

$$
u = \cos\left(x\sqrt{\frac{r'^2 - 2rr''}{4r^2}}\right), \quad v = \sin\left(x\sqrt{\frac{r'^2 - 2rr''}{4r^2}}\right), \tag{3.109}
$$

where r is given by (3.103). This give in fact u and v in terms of $C = A_2^2$. We know that the linear independent solutions to (3.101) are given by the $u^k v^{n-1-k}, 0 \leq k \leq n-1$. Solutions

(3.108) correspond to the case where $C \geq 0$ and it has to be noted that the general solutions are given by

$$y = A\exp\{\alpha x\} + B\exp\{\bar{\alpha}x\}, \tag{3.110}$$

for some constants A and B, α is the root of $x^2 + C = 0$ (C real).

3.4.5 Case $A_2^2 = \alpha x^m$, $\quad y'' + \alpha x^m y = 0$.

$m \neq -2$: Solving $y'' + \alpha x^m y = 0$ is a straightforward application of the theorem 3.4.2 for $k = 2$ and $f(x) = -\alpha x^m$. Let $f_o(x) = 1$ and $c = 0$. Then

$$f_1(x) = \int_0^x \int_0^{u_1} f_0(u)f(u)du\,du_1 + \sum_{j=0}^{1} a_j^{(1)} x^j \tag{3.111}$$

$$= -\frac{\alpha}{m+1}\int_0^x {u_1}^{m+1}du_1 \tag{3.112}$$

$$= -\frac{\alpha}{(m+1)(m+2)}x^{m+2}. \tag{3.113}$$

The expressions of $f_2(x), f_3(x), \ldots f_n(x)$ (using (3.96)) are given by

$$f_2(x) = \frac{\alpha^2}{(m+1)(m+2)(2m+3)(2m+4)}x^{2m+4} \tag{3.114}$$

$$f_3(x) = \frac{-\alpha^3}{(m+1)(m+2)(2m+3)(2m+4)(3m+5)(3m+6)}x^{3m+6} \tag{3.115}$$

$$\vdots \tag{3.116}$$

$$f_n(x) = \frac{(-\alpha)^n}{(m+1)(m+2)(2m+3)(2m+4)\ldots(nm+2n-1)(nm+2n)}x^{nm+2n} \tag{3.117}$$

Similarly, letting $g_o(x) = x$ we have

$$g_n(x) = \frac{(-\alpha)^n}{(m+2)(m+3)(2m+4)(2m+5)\ldots(nm+2n)(nm+2n+1)}x^{nm+2n} \tag{3.118}$$

By Theorem (3.4.2) ,

$$u(x) = \sum_{i=0}^{\infty} f_n(x) \quad \text{and} \quad v = \sum_{i=0}^{\infty} g_n(x) \tag{3.119}$$

are solutions to $y'' + \alpha x^m y = 0$. In terms of special functions, Mathematica gives

$$u = (m+2)^{-\frac{1}{m+2}}\sqrt{x}\,\Gamma\left(\frac{m+1}{m+2}\right) J_{-\frac{1}{m+2}}\left(\frac{2x^{\frac{m}{2}+1}}{m+2}\sqrt{\alpha}\right)\alpha^{\frac{1}{2m+4}} \tag{3.120}$$

$$v = (m+2)^{-\frac{1}{m+2}}\sqrt{x}\,\Gamma\left(1+\frac{1}{m+2}\right) J_{\frac{1}{m+2}}\left(\frac{2x^{\frac{m}{2}+1}}{m+2}\sqrt{\alpha}\right)\alpha^{\frac{1}{2m+4}}, \qquad m \neq -2, \tag{3.121}$$

where Γ is the $Gamma$ function, J is the Bessel function and $x = \left(\frac{A_2^2}{\alpha}\right)^{\frac{1}{m}}$.

Case $m = -2$ i.e. $A_2^2 = \alpha/x^2$: The corresponding equation is

$$y'' + \frac{\alpha}{x^2}y = 0, \tag{3.122}$$

which is equivalent to

$$x^2y'' + \alpha y = 0 \tag{3.123}$$

Let $x = e^t$, then

$$dx = e^t dt, \tag{3.124a}$$
$$y(x)' = e^{-t}Y(t)' \tag{3.124b}$$
$$y''(x) = Y(t)''e^{-2t} - Y(t)'e^{-2t}. \tag{3.124c}$$

Substituting back (3.124a) in (3.123) we get

$$Y'' - Y' + \alpha Y = 0, \tag{3.124d}$$

and its characteristic equation is $Q^2 - Q + \alpha = 0$.

- If $\alpha \le \frac{1}{4}$ then the solutions of (3.123) are given by

$$y_1 = x^{\frac{1-\sqrt{1-4\alpha}}{2}} \quad , \quad y_2 = x^{\frac{1+\sqrt{1-4\alpha}}{2}} \tag{3.124e}$$

- If If $\alpha \ge \frac{1}{4}$ then the solutions of (3.123) are given by

$$y_1 = \sqrt{x}\cos\left(\frac{\sqrt{4\alpha - 1}}{2}\ln x\right) \quad , \quad y_2 = \sqrt{x}\sin\left(\frac{\sqrt{4\alpha - 1}}{2}\ln x\right) \tag{3.124f}$$

- If $\alpha = \frac{1}{4}$, the solutions of (3.123) are given by

$$y_1 = \sqrt{x} \quad , \quad y_2 = \sqrt{x}\ln x \tag{3.124g}$$

with $x = \sqrt{\frac{\alpha}{A_2^2}}$.

3.4.6 Case $A_2^2 = e^{bx}$, $b \neq 0$. Equation (1.14) becomes

$$y'' + e^{bx}y = 0. \tag{3.125}$$

Setting $X = e^{bx}$ and $Y(X) \equiv y(x)$, we get $x = \frac{1}{b}\ln(X)$. Let us express (3.125) in terms of the new variables X and Y.

-
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dX}\frac{dX}{dx} \\ &= bX\,Y'\end{aligned} \tag{3.126}$$

  and

-
$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dX}\left(bX\,\frac{dY}{dX}\right)\frac{dX}{dx} \\ &= \left(b\frac{dY}{dX} + bX\,\frac{d^2Y}{dX^2}\right)\,bX \\ &= b^2XY' + b^2X^2Y''\end{aligned} \tag{3.127}$$

The substitution of (3.126) and (3.127) in (1.14) leads to

$$b^2X^2Y'' + b^2XY' + XY = 0 \tag{3.128}$$

which is equivalent to

$$b^2XY'' + b^2Y' + Y = 0. \tag{3.129}$$

Applying the Laplace Transform $\mathcal{L}[f(x)](s) = \int_0^\infty f(x)e^{-sx}dx$ to (3.129) we get

$$b^2\mathcal{L}[XY''](s) + b^2\mathcal{L}[Y'](s) + \mathcal{L}[Y](s) = \mathcal{L}[0](s). \tag{3.130}$$

Using the properties

$$\mathcal{L}[Y'](s) = s\mathcal{L}[Y](s) - Y(0) \quad , \quad \mathcal{L}[XY''](s) = -\frac{d}{ds}\left(\mathcal{L}[Y''](s)\right) \tag{3.131}$$

and

$$\mathcal{L}[Y''](s) = s^2\mathcal{L}[Y](s) - sY(0) - Y'(0), \tag{3.132}$$

equation (3.130) becomes

$$-b^2\frac{d}{ds}\left[s^2\mathcal{L}[Y](s) - sY(0) - Y'(0)\right] + b^2s\mathcal{L}[Y](s) - b^2Y(0) + \mathcal{L}[Y](s) = 0. \tag{3.133}$$

By expanding the above equation we get

$$-2b^2s\mathcal{L}[Y](s) - b^2s^2\frac{d}{ds}\mathcal{L}[Y](s) + b^2s\mathcal{L}[Y](s) - b^2Y(0) + \mathcal{L}[Y](s) = 0, \tag{3.134}$$

which is equivalent to the separable equation

$$-\,b^2s^2\left[\mathcal{L}[Y](s)\right]' + [-2b^2s + b^2s + 1]\mathcal{L}[Y](s) = 0. \tag{3.135}$$

We may rewrite (3.135) as follows

$$\frac{[\mathcal{L}[Y](s)]'}{\mathcal{L}[Y](s)} = -\frac{1}{s} + \frac{1}{b^2 s^2}. \tag{3.136}$$

The integration of (3.136) gives

$$\ln \mathcal{L}[Y](s) = -\ln s - \frac{1}{b^2 s} + k, \tag{3.137}$$

where k is an arbitrary constant. Applying the exponential function to both sides we get

$$\mathcal{L}[Y](s) = \frac{c}{s} e^{-\frac{1}{b^2 s}}. \tag{3.138}$$

The inverse Laplace transform of (3.138) gives (taking into consideration $X = e^{bx}$) the solution

$$u = J_0\left(\frac{2}{b}\sqrt{e^{bx}}\right), \tag{3.139}$$

where J_0 is the Bessel function of the first kind, thus v is the Bessel function of the second kind. Therefore the two solutions in this case are given by

$$u = J_0\left(\frac{2}{b}\sqrt{e^{bx}}\right) \quad , \quad v = Y_0\left(\frac{2}{b}\sqrt{e^{bx}}\right), \tag{3.140}$$

with $A_2^2 = e^{bx}$.

3.4.7 Reduction by symmetries.

We want to solve the equation

$$y'' + a\,y = 0. \tag{3.141}$$

Let $v = \xi\partial_x + \phi\partial_y$ be the infinitesimal generator of (3.141). Applying the invariant condition we get

$$pr^{(2)}v[y'' + ay] = 0 \tag{3.142}$$

$$(\xi\partial_x + \phi\partial_y + \phi^x\partial_{y_x} + \phi^{xx}\partial_{xx})[y'' + ay] = 0 \tag{3.143}$$

$$\xi a' y + \phi a + \phi^{xx} = 0 \tag{3.144}$$

We substitute ϕ^{xx} with its expression to get

$$\begin{aligned}&\xi a' y + \phi a + \phi_{xx} + (2\phi_{yx} - \xi_{xx})y_x + (\phi_y - 2\xi_x)y_{xx} + (\phi_{yy} - 2\xi_{yx}){y_x}^2 \\ &- \xi_{yy}y_x^3 - 3\xi_y y_x y_{xx} = 0\end{aligned} \tag{3.145}$$

The substitution of y'' with $-ay$ in (3.145) gives

$$\begin{aligned}&\phi a + \phi_{xx} + (2\phi_{yx} - \xi_{xx})y_x + [\xi a' - a(\phi_y - 2\xi_x)]y + (\phi_{yy} - 2\xi_{yx}){y_x}^2 \\ &- \xi_{yy}y_x^3 + 3a\xi_y y y_x = 0\end{aligned} \tag{3.146}$$

The system of determining equations is

$$
\begin{aligned}
1 &: \phi a + \phi_{xx} + (2a\xi_x - a\phi_y + a'\xi)y = 0 && (3.147a)\\
y_x &: 2\phi_{yx} - \xi_{xx} + 3a\xi_y y = 0 && (3.147b)\\
y_x^2 &: \phi_{yy} - 2\xi_{yx} = 0 && (3.147c)\\
y_x^3 &: \xi_{yy} = 0. && (3.147d)
\end{aligned}
$$

From (3.147d) we deduce that

$$\xi(x, y) = f(x)y + g(x), \tag{3.148}$$

where f and g are arbitrary functions. Using (3.148) in (3.147c) we get

$$\phi_{yy} = 2f'(x). \tag{3.149}$$

Then now,

$$\phi = f'(x)y^2 + h(x)y + i(x), \tag{3.150}$$

where h, i are arbitrary functions. Substituting (3.148) and (3.150) in (3.147a) and (3.147b) we get the following equations

$$
\begin{aligned}
(f^{(3)} + af' + a'f)y^2 + (h^{(2)} + 2ag' + a'g)y + (i^{(2)} + ai) &= 0 && (3.151a)\\
(3f'' + 3af)y + 2h' - g'' &= 0 && (3.151b)
\end{aligned}
$$

Separating (3.151) by y^n we get

$$
\begin{aligned}
y^2 &: f^{(3)} + af' + a'f = 0 && (3.152a)\\
y &: h^{(2)} + 2ag' + a'g = 0 && (3.152b)\\
1 &: i^{(2)} + ai = 0 && (3.152c)
\end{aligned}
$$

and

$$
\begin{aligned}
y &: f^{(2)} + af = 0 && (3.153a)\\
1 &: 2h' - g^{(2)} = 0. && (3.153b)
\end{aligned}
$$

The first equation of (3.152) is just the derivative of the first equation of (3.153). The second equation of (3.152) and (3.153) lead to

$$g^{(3)} + 4ag' + 2a'g = 0 \tag{3.154}$$

which is an iterative equation. Letting u and v be solutions of $y'' + ay = 0$, the reader can readily check that the expressions of ξ and ϕ are given as follows

$$
\begin{aligned}
\xi(x, y) &= (\alpha_6 u + \alpha_7 v)y + \alpha_1 u^2 + \alpha_2 v^2 + \alpha_3 uv && (3.155a)\\
\phi(x, y) &= (\alpha_6 u' + \alpha_7 v')y^2 + \left(\alpha_1 uu' + \alpha_2 vv' + \frac{\alpha_3}{2}(u'v + uv') + c_0\right) y + \alpha_4 u + \alpha_5 v. && (3.155b)
\end{aligned}
$$

Therefore, the eight vectors that spanned the lie algebra are given by

$$v_0 = y\partial_y \tag{3.156a}$$
$$v_1 = u^2\partial_x + uu'y\partial_y \tag{3.156b}$$
$$v_2 = v^2\partial_x + vv'y\partial_y \tag{3.156c}$$
$$v_3 = uv\partial_x + \frac{1}{2}(u'v + uv')y\partial_y \tag{3.156d}$$
$$v_4 = u\partial_y \tag{3.156e}$$
$$v_5 = v\partial_y \tag{3.156f}$$
$$v_6 = uy\partial_x + u'y^2\partial_y \tag{3.156g}$$
$$v_7 = vy\partial_x + v'y^2\partial_y. \tag{3.156h}$$

Consider the symmetry $v_0 = y\partial_y$: (3.141) is invariant under the group of transformations

$$x^* = x \qquad\qquad y^* = \lambda y. \tag{3.157}$$

The differential invariants can be found by solving the characteristic equation

$$\frac{dx}{0} = \frac{dy}{y}. \tag{3.158}$$

Hence, the two invariants are given by $z = x$ and $w = \log y$. The first and second derivatives of y with respect to x are given by

$$y' = w_z e^w \qquad\qquad y'' = w_{zz}e^w + w_z^2 e^w. \tag{3.159}$$

Using these expressions in (3.141) leads to

$$w_{zz}e^w + w_z^2 e^w + ae^w = 0, \tag{3.160}$$

which can be written as

$$w_{zz} + w_z^2 + a = 0. \tag{3.161}$$

Setting $W = w_z$ in the above equation yields

$$W_z = -W^2 - a \tag{3.162}$$

which is known as a Riccati equation. Let W_p be a particular solution of (3.162). By setting $T = \dfrac{1}{W - W_p}$, we then have $T_z = \dfrac{W + W_p}{W - W_p}$. The substitution of W with its expression in terms of T in (3.162) gives

$$T_z - 2W_pT = 1 \tag{3.163}$$

and has

$$T = k_1 \exp\left(\int 2W_p\right) - \frac{1}{2W_p} \tag{3.164}$$

as solution. We just need to inverse all the transformations to retrieve y .

3.5 Point transformations

To find all the mappings we are going to solve (2.42) for $x^\star$ and $y^\star$ in each case. We said that the $n+4$ symmetries of

$$y^{(n)} + \sum_{i=0}^{n-2} a_i \, y^{(i)} = 0 \tag{3.165}$$

are given by

$$v_0 = y\partial_y \tag{3.166a}$$
$$v_1 = u^2\partial_x + (n-1)uu'y\partial_y \tag{3.166b}$$
$$v_2 = uv\partial_x + \frac{n-1}{2}(u'v + uv')y\partial_y \tag{3.166c}$$
$$v_3 = v^2\partial_x + (n-1)vv'y\partial_y \tag{3.166d}$$
$$v_k = u^{n-1-k}v^k\partial_y \quad , k = 4, \dots n+3, \tag{3.166e}$$

where u and v are the solutions of the second-order source equation. Vectors v_1, v_2, v_3 can be represented by

$$V_{123} = A(x)\partial_x + B(x)y\partial_y \tag{3.167}$$

for some functions A and B of x, since u and v are functions of x. We found that v_0 leads to the point transformations

$$x^\star = x, \tag{3.168a}$$
$$y^\star = e^\varepsilon y. \tag{3.168b}$$

The group of transformation (3.168) transforms (3.165) into

$$e^\varepsilon \left(y^{(n)} + A_n^2(x)y^{(n-2)} + \dots + A_n^n(x)y \right) = 0 \tag{3.169}$$

which is the same as (3.165) since e^ε can not be zero. The symmetry v_k generates the group of transformations

$$x^\star = x, \tag{3.170a}$$
$$y^\star = u^{n-1-k}v^k\varepsilon + y \tag{3.170b}$$

that transform (3.165) into

$$\begin{aligned} &\left(y^{(n)} + A_n^2(x)y^{(n-2)} + \dots + A_n^n(x)y\right) + \varepsilon\Big[\left(u^{n-1-k}v^k\right)^{(n)} + A_n^2(x)\left(u^{n-1-k}v^k\right)^{(n-2)} \\ &+ \dots + A_n^n(x)\left(u^{n-1-k}v^k\right) \Big] = 0. \end{aligned} \tag{3.171}$$

Given that $u^{n-1-k}v^k$ are solutions of (3.165) we end up with an unchanged equation. Using the vector $v_{123} = f(x)\partial_x + g(x)y\partial_y$ we get

$$\frac{\partial x^\star}{\partial \varepsilon} = f(x^\star), \qquad \frac{\partial y^\star}{\partial \varepsilon} = g(x^\star)y^\star. \tag{3.172}$$

Then

$$\frac{\partial x^\star}{\partial \varepsilon} = f(x^\star) \Rightarrow x^\star = \tilde{f}(x,\varepsilon) \tag{3.173a}$$

and

$$\frac{\partial y^\star}{\partial \varepsilon} = g(x^\star)y^\star \Rightarrow y^\star = \tilde{g}(x,\varepsilon)y \tag{3.173b}$$

give the most general form of the group of symmetry transformations generated by V_{123}:

$$x^\star = \tilde{f}(x,\varepsilon) \tag{3.174a}$$
$$y^\star = \tilde{g}(x,\varepsilon)y. \tag{3.174b}$$

3.5.1 Example. In this example we aim to find all the point transformations that map solutions onto others solutions for $r = c_1 x + c_2$. The $n+4$ vectors that spanned the Lie algebra in this special case are given by

$$v_0 = y\partial_y \tag{3.175a}$$
$$v_1 = r\partial_x - sy\partial_y \tag{3.175b}$$
$$v_2 = r\ln r\partial_x - (1+\ln r)sy\partial_y \tag{3.175c}$$
$$v_3 = r\ln^2 r\partial_x - \left(\ln^2 r + 2\ln r\right)sy\partial_y \tag{3.175d}$$
$$v_k = r^{\frac{n-1}{2}}\ln^k r\partial_y \quad ,k = 4,\dots n+3. \tag{3.175e}$$

- For $v_0 = y\partial_y$, the infinitesimals are given by

$$\xi = 0, \qquad \phi = y, \tag{3.176}$$

then, we have

$$\frac{\partial x^\star}{\partial \varepsilon} = 0, \qquad \frac{\partial y^\star}{\partial \varepsilon} = y^\star \tag{3.177}$$

$$\frac{\partial x^\star}{\partial \varepsilon} = 0 \qquad \Longrightarrow \qquad x^\star = cst. \tag{3.178}$$

Using condition (2.42), the value of this constant is equal to x, then,

$$x^\star = x \tag{3.179}$$

Also,

$$\frac{\partial y^\star}{\partial \varepsilon} = y^\star \qquad \Longrightarrow \qquad \ln y^\star = \varepsilon + c_2. \tag{3.180}$$

Conditions (2.42) imply that $c_2 = \ln y$ then,

$$y^\star = e^\varepsilon y. \tag{3.181}$$

Therefore, the point transformations obtained from v_0 are

$$x^\star = x, \tag{3.182}$$
$$y^\star = e^\varepsilon y. \tag{3.183}$$

- For $v_1 = (c_1 x + c_2)\partial_x + \dfrac{n-1}{2} c_1 y \partial_y$.

$$\xi = c_1 x + c_2, \qquad \phi = \frac{n-1}{2} c_1 y, \tag{3.184}$$

and

$$\frac{\partial x^\star}{\partial \varepsilon} = c_1 x^\star + c_2, \qquad \frac{\partial y^\star}{\partial \varepsilon} = \frac{n-1}{2} c_1 y^\star. \tag{3.185}$$

Taking into consideration conditions (2.42), we get

$$c_1 x^\star + c_2 = (c_1 x + c_2) \exp(c_1 \varepsilon), \tag{3.186}$$

$$y^\star = y \exp\left(\frac{n-1}{2} c_1 \varepsilon\right). \tag{3.187}$$

- For $v_2 = r \ln r \, \partial_x - (1 + \ln r)\, sy \, \partial_y$. Here,

$$\xi = (c_1 x + c_2) \ln(c_1 x + c_2), \qquad \phi = \frac{n-1}{2} c_1 \left[1 + \ln(c_1 x + c_2)\right] y, \tag{3.188}$$

then,

$$\frac{\partial x^\star}{\partial \varepsilon} = (c_1 x^\star + c_2) \ln(c_1 x^\star + c_2), \qquad \frac{\partial y^\star}{\partial \varepsilon} = \frac{n-1}{2} c_1 \left[1 + \ln(c_1 x^\star + c_2)\right] y^\star. \tag{3.189}$$

We have

$$\frac{\partial x^\star}{\partial \varepsilon} = (c_1 x^\star + c_2) \ln(c_1 x^\star + c_2), \qquad \Rightarrow \qquad \ln\left[\ln(c_1 x^\star + c_2)\right] = c_1 \varepsilon + c. \tag{3.190}$$

From (2.42) we have $c = \ln\left[\ln(c_1 x + c_2)\right]$. Substituting the expression of c in the right hand side expression of (3.190) and applying the exponential function twice we get

$$c_1 x^\star + c_2 = (c_1 x + c_2)^{\exp(c_1 \varepsilon)}. \tag{3.191}$$

Using (3.191) in

$$\frac{\partial y^\star}{\partial \varepsilon} = \frac{n-1}{2} c_1 \left[1 + \ln(c_1 x^\star + c_2)\right] y^\star \tag{3.192}$$

yields

$$\frac{\partial y^\star}{\partial \varepsilon} = \frac{n-1}{2} c_1 \left[1 + \exp(c_1 \varepsilon) \ln(c_1 x + c_2)\right] y^\star. \tag{3.193}$$

Solving for $y^\star$ and taking into consideration (2.42) we get

$$y^\star = \left[\exp(c_1 \varepsilon)(c_1 x + c_2)^{(\exp(c_1 \varepsilon) - 1)}\right]^{\frac{n-1}{2}} y. \tag{3.194}$$

Therefore, the point transformations obtained from v_2 are

$$c_1 x^\star + c_2 = (c_1 x + c_2)^{\exp(c_1 \varepsilon)} \tag{3.195}$$

$$y^\star = \left[\exp(c_1 \varepsilon)(c_1 x + c_2)^{(\exp(c_1 \varepsilon) - 1)}\right]^{\frac{n-1}{2}} y. \tag{3.196}$$

- For $v_3 = r\ln^2 r\partial_x - \left(\ln^2 r + 2\ln r\right) sy\partial_y$, we have

$$\xi = (c_1x + c_2)\ln^2(c_1x + c_2), \qquad \phi = (n-1)c_1\left[\frac{1}{2}\ln^2(c_1x + c_2) + \ln(c_1x + c_2)\right] y, \tag{3.197}$$

then,

$$\frac{\partial x^\star}{\partial \varepsilon} = (c_1x^\star + c_2)\ln^2(c_1x^\star + c_2), \quad \frac{\partial y^\star}{\partial \varepsilon} = (n-1)c_1\left[\frac{1}{2}\ln^2(c_1x^\star + c_2) + \ln(c_1x^\star + c_2)\right] y^\star. \tag{3.198}$$

Thus,

$$\frac{\partial x^\star}{\partial \varepsilon} = (c_1x^\star + c_2)\ln^2(c_1x^\star + c_2), \qquad \Rightarrow \qquad -\frac{1}{\ln(c_1x^\star + c_2)} = c_1\varepsilon + c. \tag{3.199}$$

From (2.42) we have $c = -\dfrac{1}{\ln(c_1x + c_2)}$. Substituting the expression of c in the right hand side expression of (3.199) and applying the exponential function we obtain

$$c_1x^\star + c_2 = (c_1x + c_2)^{\frac{1}{1-c_1\varepsilon\ln(c_1x+c_2)}}. \tag{3.200}$$

Using (3.200) in

$$\frac{\partial y^\star}{\partial \varepsilon} = (n-1)c_1\left[\frac{1}{2}\ln^2(c_1x^\star + c_2) + \ln(c_1x^\star + c_2)\right] y^\star \tag{3.201}$$

yields

$$\frac{\partial y^\star}{y^\star} = (n-1)\left[\left(\frac{\ln(c_1x + c_2)}{2}\right)\left(\frac{c_1\ln(c_1x + c_2)}{(1 - c_1\varepsilon\ln(c_1x + c_2))^2}\right) + \frac{c_1\ln(c_1x + c_2)}{1 - c_1\varepsilon\ln(c_1x + c_2)}\right]\partial\varepsilon. \tag{3.202}$$

Integrating the above equation, we get

$$\ln y^\star = (n-1)\left(\frac{\ln(c_1x + c_2)}{2\left[1 - c_1\varepsilon\ln(c_1x + c_2)\right]} + \ln\left[\frac{1}{1 - c_1\varepsilon\ln(c_1x + c_2)}\right]\right) + cst. \tag{3.203}$$

Solving for $y^\star$ (taking into consideration (2.42)), we have

$$y^\star = \left[\frac{(c_1x + c_2)^{\frac{c_1\varepsilon\ln(c_1x+c_2)}{2[1-c_1\varepsilon\ln(c_1x+c_2)]}}}{1 - c_1\varepsilon\ln(c_1x + c_2)}\right]^{n-1} y. \tag{3.204}$$

The point transformations obtained from v_3 are

$$c_1x^\star + c_2 = (c_1x + c_2)^{\frac{1}{1-c_1\varepsilon\ln(c_1x+c_2)}} \tag{3.205}$$

$$y^\star = \left[\frac{(c_1x + c_2)^{\frac{c_1\varepsilon\ln(c_1x+c_2)}{2[1-c_1\varepsilon\ln(c_1x+c_2)]}}}{1 - c_1\varepsilon\ln(c_1x + c_2)}\right]^{n-1} y. \tag{3.206}$$

- For $v_k = r^{\frac{n-1}{2}} \ln^k r\, \partial_y \quad , k = 4, \dots n+3$, the point transformations are given by

$$x^\star = x \tag{3.207}$$

$$y^\star = y + (c_1 x + c_2)^{\frac{n-1}{2}} \ln^k (c_1 x + c_2)\varepsilon. \tag{3.208}$$

List of point transformations obtained for some given values of r.

(a) Case $r =$ constant.

$$(x^\star, y^\star) = (x, e^\varepsilon y)\,, \quad (x^\star, y^\star) = (x+\varepsilon, y)\,, \quad (x^\star, y^\star) = \left(e^\varepsilon x, e^{\frac{n-1}{2}\varepsilon} y\right), \tag{3.209a}$$

$$(x^\star, y^\star) = \left(\frac{x}{1-\varepsilon x}, \left(\frac{1}{1-x\varepsilon}\right)^{n-1} y\right), \quad \left(x^\star, y^\star = y + x^k \varepsilon\right). \tag{3.209b}$$

(b) Case $r = (c_1 x + c_2)^2$.

$$(x^\star, y^\star) = (x, e^\varepsilon y)\,, \quad (x^\star, y^\star) = (x+\varepsilon, y)\,, \tag{3.210a}$$

$$(c_1 x^\star + c_2, y^\star) = \left(e^{c_1 \varepsilon}(c_1 x + c_2), \exp \frac{n-1}{2} c_1 \varepsilon y\right), \tag{3.210b}$$

$$(c_1 x^\star + c_2, y^\star) = \left(\frac{c_1 x + c_2}{1 - c_1 \varepsilon (c_1 x + c_2)}, \left(\frac{1}{1 - c_1 (c_1 x + c_2)\varepsilon}\right)^{n-1} y\right), \tag{3.210c}$$

$$(x^\star, y^\star) = \left(x, y + (c_1 x + c_2)^k \varepsilon\right) \tag{3.210d}$$

(c) Case $r = (c_1 x + c_2)^m$

$$(x^\star, y^\star) = (x, e^\varepsilon y)\,, \tag{3.211a}$$

$$\begin{cases} (c_1 x^\star + c_2)^{m-1} = \dfrac{(c_1 x + c_2)^{m-1}}{1 - c_1(m-1)(c_1 x + c_2)^{m-1}\varepsilon}, \\ y^\star = \left(\dfrac{1}{1 - c_1(m-1)(c_1 x + c_2)^{m-1}\varepsilon}\right)^{\frac{m(n-1)}{2(m-1)}} y, \end{cases}$$

$$(c_1 x^\star + c_2, y^\star) = \left((c_1 x + c_2)\exp(c_1 \varepsilon)\,, y \exp\left(\frac{n-1}{2} c_1 \varepsilon\right)\right), \tag{3.2}$$

$$\begin{cases} (c_1 x^\star + c_2)^{1-m} = \dfrac{(c_1 x + c_2)^{1-m}}{1 - c_1(1-m)(c_1 x + c_2)^{1-m}\varepsilon}, \\ y^\star = \left(\dfrac{1}{1 - c_1(1-m)(c_1 x + c_2)^{1-m}\varepsilon}\right)^{\frac{(2-m)(n-1)}{2(1-m)}} y, \end{cases}$$

$$(x^\star, y^\star) = \left(x, y + (c_1 x + c_2)^{\frac{1}{2}[2k + m(n-1-2k)]}\varepsilon\right). \tag{3.3}$$

Case $r = e^{mx}$

$$(x^\star, y^\star) = (x, e^\varepsilon y)\,, \quad \left(e^{mx^\star}, y^\star\right) = \left(\frac{e^{mx}}{1 - m\varepsilon e^{mx}}, \left(\frac{1}{1 - m\varepsilon e^{mx}}\right)^{\frac{n-1}{2}\varepsilon} y\right), \tag{3.212a}$$

$$\left(e^{-mx^\star}, y^\star\right) = \left(\frac{e^{-mx}}{1 + m\varepsilon e^{-mx}}, \left(\frac{1}{1 + m\varepsilon e^{-mx}}\right)^{\frac{n-1}{2}\varepsilon} y\right), \tag{3.212b}$$

$$(x^\star, y^\star) = \left(x, y + \exp\left(\frac{n-1-2k}{2}mx\right)\varepsilon\right). \tag{3.212c}$$

3.6 Conclusion

We determined all the Lie point symmetries of the second-, third- and fourth-order linear iterative equation in normal form and we have made contribution to the work in the papers [1, 13] by reducing the condition on the infinitesimal f (see (3.35a)) and by expressing the infinitesimal generator in terms of the parameters r and s of the first-order source equation for a linear iterative equation of order n, $n \geq 3$. By letting Ω_n be the linear operator corresponding to the linear iterative equation of order n with source equation $y'' + ay = 0$, we showed that Ω_n, $n \geq 3$, satisfy equation (3.51). A list of linear iterative equations of order up to 10, in both normal and standard forms that have constant coefficient was given, and the point transformations needed to retrieve other solutions were provided.

4. Parameters of the transformed equation under equivalence transformations

4.1 Introduction

The group of equivalence transformations G of a family $\mathcal{A}$ of differential equations of a specified form and labelled by a set of arbitrary functions is the largest group of invertible point transformations that map each element of $\mathcal{A}$ to another element of $\mathcal{A}$. On the other hand, we know that two equations are said to be equivalent if they can be mapped to each other by an invertible point transformation. In this chapter, we shall be interested in finding the parameters of the source equation for the transformed equation under an equivalence transformation of a given iterative equation.

4.2 Equivalence transformations

Let us consider the linear equation

$$y^{(n)} + A_n^1(x)y^{(n-1)} + A_n^2(x)y^{(n-2)} + \cdots + A_n^n(x)y = 0, \tag{4.1}$$

where x is the independent variable and y the dependent variable. If we suppose that

$$\begin{cases} x = \alpha(z, w) \\ y = \beta(z, w) \end{cases} \tag{4.2}$$

is an equivalence transformation mapping (4.1) to an equivalent equation then the latter must have the same form as (4.1). The substitution of x and y in terms of new variables z and w in (4.1) must yields an equation of the form

$$w^{(n)} + B_n^1(z)\, w^{(n-1)} + B_n^2(z)\, w^{(n-2)} + \cdots + B_n^n(z)\, w = 0, \tag{4.3}$$

where z is the independent variable and w is the dependent variable. However, it is well-known [6, 12, 13] that the group of equivalence transformations of the general linear equation in standard form (4.1) is given by transformations of the form

$$\begin{cases} x = f(z) \\ y = g(z)w, \end{cases} \tag{4.4}$$

where f and g are arbitrary functions.

4.2.1 Example. Let

$$y'' + ay = 0 \tag{4.5}$$

be a linear equation of order two and since (4.5) is a special case of (4.1) its equivalent transformations can be sought into the specific form (4.4). From (4.4) we know that $dx = f'dz$ therefore, the first and second derivatives of y in terms of the news variables are given by

$$y' = \frac{g}{f'}w' + \frac{g'}{f'}w \tag{4.6}$$

$$y'' = \frac{g}{f'^2}w'' + \left(\frac{2f'g' - f''g}{f'^3}\right)w' + \left(\frac{f'g'' - f''g'}{f'^3}\right)w \tag{4.7}$$

Substituting $y = wg$ and using (4.7) into (4.5) leads to

$$\frac{g}{f'^2}w'' + \left(\frac{2f'g' - f''g}{f'^3}\right)w' + \left(\frac{f'g'' - f''g'}{f'^3} + ag\right)w = 0, \tag{4.8}$$

which can be also written as

$$w'' + \left(\frac{2f'g' - f''g}{f'g}\right)w' + \left(\frac{f'g'' - f''g'}{f'g} + (a \circ f)f'^2\right)w = 0, \tag{4.9}$$

where $\circ$ denotes the composition of functions. Transformation (4.4) defines a group of equivalence transformations of (4.5) if and only if

$$\frac{2f'g' - f''g}{f'g} = 0 \qquad \text{i.e} \qquad g = k\sqrt{f'}, \tag{4.10}$$

where k is a given constant. Hence the transformation

$$\begin{cases} x = f(z) \\ y = k\sqrt{f'(z)}\, w(z), \end{cases}$$

defines a group of equivalence transformations of (4.5).

Moreover, we note that by assuming the equation (4.1) to be in its normal form, (4.4) reduces to

$$x = f(z), \qquad y = \lambda\, [f'(z)]^{\frac{n-1}{2}}\, w, \tag{4.11}$$

where λ is an arbitrary constant while f is an arbitrary function.

Also note that a symmetry group transforms the differential equation into the same equation. So, the transformed equation of (4.1) will then be of the form

$$w^{(n)} + A_n^1(z)w^{(n-1)} + A_n^2(z)w^{(n-2)} + \cdots + A_n^n(z)w = 0 \tag{4.12}$$

under a symmetry group. We can say that a symmetry transformation is a special case of an equivalence transformation because it preserves not only the form but also the equation itself, as it leaves the equation locally unchanged.

In the previous example (4.2.1), we have a symmetry transformation for $f(z) = z + c$, where c is an arbitrary constant. As we already mentioned, the symmetry algebra of two given equivalent equations are isomorphic, and thus if one of them has maximal dimension, the other one will also be of maximal dimension, but having maximal dimension is equivalent to being iterative. Therefore under an equivalence transformation an iterative equation remains iterative. In the next section, we shall be interested in finding the parameters of the source equation for the transformed equation under an equivalence transformation of a given iterative equation.

4.3 Parameters of the transformed equations

Consider the linear iterative equation in the standard form

$$\Psi^n y \equiv K_n^0\, y^{(n)} + K_n^1\, y^{(n-1)} + K_n^2\, y^{(n-2)} + \cdots + K_n^{n-1}\, y' + K_n^n\, y = 0 \tag{4.13}$$

and let

$$y^{(n)} + A_n^2\, y^{(n-2)} + \cdots + A_n^j\, y^{(n-j)} + \cdots + A_n^n\, y = 0 \tag{4.14}$$

be the normal reduced form of (4.13). Suppose that equation (4.13), which may be written again as

$$\Delta_n(y) \equiv \Psi^n y = 0 \tag{4.15}$$

has first-order source equation

$$r(x)y' + s(x)y \equiv \Psi(y). \tag{4.16}$$

Let

$$\Omega_n(w) \equiv \Phi^n w = 0 \tag{4.17}$$

be an equivalent equation with source equation

$$R(z)w' + S(z)w = \Phi(w) \tag{4.18}$$

obtained from $\Delta_n(y) = 0$ by the transformations (4.4). We may assume that

$$\Phi^n(w) \equiv Z_n^0\, w^{(n)} + Z_n^1\, w^{(n-1)} + Z_n^2\, w^{(n-2)} + \cdots + Z_n^{n-1}\, w' + Z_n^n\, w = 0 \tag{4.19}$$

and let

$$w^{(n)} + B_n^2\, w^{(n-2)} + \cdots + B_n^{n-1}\, w' + B_n^n\, w = 0 \tag{4.20}$$

be its normal reduced form. We want to find out the parameters R and S of the first-order source equation of the transformed equation $R(z)w' + S(z)w = \Phi(w)$ in terms of the parameters r, s defined in (4.16). The idea consists of iterating with the operator Ψ n times and transforming the resulting equation with (4.4) and call it $D(z, w, n)$. On the other hand, use the operator Φ to iterate directly n times. We now find a relationship between A_n^j and B_n^j in terms of f and g (parameters of the equivalent transformation).

4.3.1 $n = 2$.

$$\Psi^{(2)}y \equiv r^2y'' + (rr' + 2rs)y' + (rs' + s^2)y \tag{4.21}$$

From (4.4) we have

$$\begin{aligned} dx &= f'(z)dz && (4.22a)\\ dy &= wg'(z)dz + g(z)w'dz && (4.22b) \end{aligned}$$

and the ratio of (4.22a) to (4.22b) gives

$$y' = \frac{dy}{dx} = \frac{g'}{f'}w + \frac{g}{f'}w. \tag{4.23}$$

So

$$\begin{aligned} y'' &= \frac{dy'}{dx}\\ &= \frac{1}{f'(z)}\frac{d}{dx}\left(\frac{g'w' + g'w}{f'}\right)\\ &= \frac{1}{f'(z)}\frac{d}{dx}\left[\frac{(g''w + g'w' + g'w' + gw'')f' - f''(g'w + gw')}{f'^2}\right]\\ &= \frac{1}{f'^3(z)}\left[f'gw'' + (2f'g' - f''g)w' + (f'g'' - f''g')w\right] && (4.24) \end{aligned}$$

According to (4.23) and (4.24), one can rewrite (4.21) as follows

$$\begin{aligned} &\frac{r^2}{f'^3}\left[f'gw'' + (2f'g' - f''g)w' + (f'g'' - f''g')w\right]\\ &+ (rr' + 2rs)\left(\frac{g}{f'}w' + \frac{g'}{f'}w\right) + (rs' + s^2)gw \equiv \Psi^{(2)}y \end{aligned} \tag{4.25}$$

Gathering all the coefficients of w, w' and w'' respectively, this gives

$$\begin{aligned} D(z, w, 2) =& \frac{(r \circ f)^2 g}{f'^2}w'' + \Big[\frac{(rof)^2(2f'g' - f''g)}{f'^3}\\ &+ \frac{(rof)(r' \circ f)f'g + 2(r \circ f)(s \circ f)g}{f'}\Big]w' + \Big[\frac{(rof)^2(f'g'' - f''g')}{f'^3} +\\ &\frac{2(rof)(s \circ f)g' + (r \circ f)(r' \circ f)f'g'}{f'} + g(r \circ f)(s' \circ f)f' + (s \circ f)^2 g\Big]w \end{aligned} \tag{4.26a}$$

which is equivalent to

$$\begin{aligned} D(z, w, 2) =& K_2^0\frac{g}{f'^2}w'' + \left[K_2^0\frac{(2f'g' - f''g)}{f'^3} + K_2^1\frac{g}{f'}\right]w'\\ &+ \left[K_2^0\frac{(f'g'' - f''g')}{f'^3} + K_2^1\frac{g'}{f'} + K_2^2 g\right]w. \end{aligned} \tag{4.26b}$$

Setting $n = 2$ in (4.19), the transformed equation becomes

$$Z_2^0 w'' + Z_2^1 w' + Z_2^2 w = 0 \tag{4.27}$$

and the substitution of $Z_2^0 w''$ with $-Z_2^1 w' - Z_2^2 w$ in (4.26b) leads to

$$\begin{aligned}&\left[-K_2^0 Z_2^1 \frac{g}{f'^2} + K_2^0 Z_2^0 \Big(\frac{(2f'g' - f''g)}{f'^3}\Big) + K_2^1 Z_2^0 \frac{g}{f'}\right] w' \\ &+ \left[-K_2^0 Z_2^2 \frac{g}{f'^2} + K_2^0 Z_2^0 \frac{(f'g'' - f''g')}{f'^3} + K_2^1 Z_2^0 \frac{g'}{f'} + K_2^2 Z_2^0 g\right] w = 0.\end{aligned} \tag{4.28}$$

The functions K_m^n, Z_o^p, f and g do not depend on w and w' therefore we can equate the coefficients of the latter to zero to yield

$$K_2^0 Z_2^1 = Z_2^0 K_2^1 f' + \frac{2f'g' - f''g}{f'g} K_2^0 Z_2^0 \tag{4.29a}$$

$$K_2^0 Z_2^2 = f'^2 K_2^2 Z_2^0 + \frac{f'g'}{g} Z_2^0 K_2^1 + \frac{f'g'' - f''g'}{f'g} K_2^0 Z_2^0. \tag{4.29b}$$

Setting

$$\mathbb{K}_n^j = \frac{K_n^j}{K_n^0}, \qquad \mathbb{Z}_n^j = \frac{Z_n^j}{Z_n^0}, \tag{4.30}$$

we deduce from equation (4.29) that

$$\mathbb{Z}_2^1 = f'\mathbb{K}_2^1 + \frac{2f'g' - f''g}{f'g} \tag{4.31a}$$

$$\mathbb{Z}_2^2 = f'^2 \mathbb{K}_2^2 + \frac{f'g'}{g}\mathbb{K}_2^1 + \frac{f'g'' - f''g'}{f'g}. \tag{4.31b}$$

4.3.2 $n = 3$**.** According to the definition of iterative equation,

$$\Psi^{(3)} y = \Psi[\Psi^{(2)} y] \tag{4.32}$$

$$\begin{aligned}&= r\frac{d}{dx}\left[r^2 y'' + (rr' + 2rs)y' + (rs' + s^2)y\right] \\ &\quad + s\left[r^2 y'' + (rr' + 2rs)y' + (rs' + s^2)\right]\end{aligned} \tag{4.33}$$

$$\begin{aligned}&= r[2rr'y'' + r^2 y^{(3)} + (r'^2 + rr'' + 2r's + 2rs')y' \\ &\quad + (rr' + 2rs)y'' + (r's' + rs'' + 2ss')y + (rs' + s^2)y'] \\ &\quad + sr^2 y'' + (rr's + 2rs^2)y' + (rss' + s^3)y\end{aligned} \tag{4.34}$$

$$\begin{aligned}\implies \Psi^{(3)} y =& r^3 y^{(3)} + (3r^2 r' + 2r^2 s)y'' + (3rr's + 3r^2 s' + rr'^2 + r^2 r'')y' \\ &+ (3rss' + r^2 s'' + rr's')y.\end{aligned} \tag{4.35}$$

For the expression of $y^{(3)}$ in terms of z and w, we first note that

$$y^{(3)} = \frac{d}{dx}(y''). \tag{4.36}$$

Using (4.24) in (4.36) we have

$$
\begin{aligned}
y^{(3)} =& \frac{1}{f'(z)}\Big[-\frac{3f''}{f'^4}\big(f'gw''+2f'g'w'-f''w'g+f'g''w-f''g'w\big)\\
&+\frac{1}{f'^3}\big(f''gw''+f'g'w''+f'gw'''+2f''g'w'+2f'g''w'+2f'g'w''\\
&-f'''gw'-f''gw''-f''g'w'+f''g''w+f'g'''w+f'g''w'\\
&-f'''g'w-f''g''w-f''g'w'\big)\Big].
\end{aligned}
\tag{4.37}
$$

Therefore,

$$
\begin{aligned}
y^{(3)} =& \left(\frac{g}{f'^3}\right)w^{(3)}+\left(-\frac{3f''g}{f'^4}+\frac{3g'}{f'^3}\right)w^{(2)}+\Big(-\frac{6f''g'}{f'^4}+\frac{3f''^2g}{f'^5}+\frac{3g''}{f'^3}\\
&-\frac{f'''g}{f'^4}\Big)w'+\left(-\frac{3f''g''}{f'^4}+\frac{3f''^2g'}{f'^5}+\frac{g'''}{f'^3}-\frac{f'''g'}{f'^4}\right)w
\end{aligned}
\tag{4.38}
$$

We substitute (4.24) and (4.38) in (4.35) to have

$$
\begin{aligned}
\Psi^{(3)}y \equiv& K_3^0\Big[\left(\frac{g}{f'^3}\right)w^{(3)}+\left(-\frac{3f''g}{f'^4}+\frac{3g'}{f'^3}\right)w^{(2)}+\Big(-\frac{6f''g'}{f'^4}\\
&+\frac{3f''^2g}{f'^5}+\frac{3g''}{f'^3}-\frac{f'''g}{f'^4}\Big)w'+\left(-\frac{3f''g''}{f'^4}+\frac{3f''^2g'}{f'^5}+\frac{g'''}{f'^3}-\frac{f'''g'}{f'^4}\right)w\Big]\\
&+K_3^1\Big[\frac{g}{f'^2}w''+\frac{2f'g'-f''g}{f'^3}w'+\frac{f'g''-f''g'}{f'^3}w\Big]+K_3^2\Big[\frac{g'}{f'}w+\frac{g}{f'}w'\Big]\\
&+K_3^3gw
\end{aligned}
\tag{4.39}
$$

Hence,

$$
\begin{aligned}
D(z,w,3) =& \left[\frac{K_3^0g}{f'^3}\right]w^{(3)}+\left[K_3^0\left(-\frac{3f''g}{f'^4}+\frac{3g'}{f'^3}\right)+K_3^1\frac{g}{f'^2}\right]w^{(2)}+\\
&\Big[K_3^0\Big(\frac{-6f''g'}{f'^4}+\frac{3f''^2g}{f'^5}+\frac{3g''}{f'^3}-\frac{f'''g}{f'^4}\Big)+K_3^1\left(\frac{2f'g'-f''g}{f'^3}\right)\\
&+K_3^2\frac{g}{f'}\Big]w'+\Big[K_3^0\left(-\frac{3f''g''}{f'^4}+\frac{3f''^2g'}{f'^5}+\frac{g'''}{f'^3}-\frac{f'''g'}{f'^4}\right)\\
&+K_3^1\left(\frac{f'g''-f''g'}{f'^3}\right)+K_3^2\frac{g'}{f'}+K_3^3g\Big]w.
\end{aligned}
\tag{4.40}
$$

We know from (4.19) that

$$
Z_3^0w^{(3)} = -(Z_3^1w^{(2)}+Z_3^2w^{(1)}+Z_3^3w)
\tag{4.41}
$$

for $n=3$. Substituting (4.41) in (4.40) and rearranging it, leads to

$$\begin{aligned}
&\Big[-K_3^0Z_3^1\left(\frac{g}{f'^3}\right)+K_3^0Z_3^0\left(-\frac{3f''g}{f'^4}+\frac{3g'}{f'^3}\right)+K_3^1Z_3^0\frac{g}{f'^2}\Big]w^{(2)}\\
&+\Big[-K_3^0Z_3^2\left(\frac{g}{f'^3}\right)+K_3^0Z_3^0\Big(-\frac{6f''g'}{f'^4}+\frac{3f''^2g}{f'^5}+\frac{3g''}{f'^3}\\
&-\frac{f'''g}{f'^4}\Big)+K_3^1Z_3^0(\frac{2f'g'-f''g}{f'^3})+Z_3^0K_3^2\frac{g}{f'}\Big]w^{(1)}+\Big[-K_3^0Z_3^3\left(\frac{g}{f'^3}\right)\\
&+K_3^0Z_3^0\left(-\frac{3f''g''}{f'^4}+\frac{3f''^2g'}{f'^5}+\frac{g'''}{f'^3}-\frac{f'''g'}{f'^4}\right)+K_3^1Z_3^0\left(\frac{f'g''-f''g'}{f'^3}\right)\\
&+Z_3^0K_3^2\frac{g'}{f'}+Z_3^0K_3^3g\Big]w=0.
\end{aligned} \tag{4.42}$$

Letting all the coefficients w, w', w'' equal to zero leads to the relationship that we are looking for:

$$-K_3^0Z_3^1\left(\frac{g}{f'^3}\right)+K_3^0Z_3^0\left(-\frac{3f''g}{f'^4}+\frac{3g'}{f'^3}\right)+K_3^1Z_3^0\frac{g}{f'^2}=0 \tag{4.43a}$$

$$\begin{aligned}
&-K_3^0Z_3^2\left(\frac{g}{f'^3}\right)+K_3^0Z_3^0\Big(-\frac{6f''g'}{f'^4}+\frac{3f''^2g}{f'^5}+\frac{3g''}{f'^3}\\
&-\frac{f'''g}{f'^4}\Big)+K_3^1Z_3^0(\frac{2f'g'-f''g}{f'^3})+Z_3^0K_3^2\frac{g}{f'}=0
\end{aligned} \tag{4.43b}$$

$$\begin{aligned}
&-K_3^0Z_3^3\left(\frac{g}{f'^3}\right)+K_3^0Z_3^0\left(-\frac{3f''g''}{f'^4}+\frac{3f''^2g'}{f'^5}+\frac{g'''}{f'^3}-\frac{f'''g'}{f'^4}\right)\\
&+K_3^1Z_3^0\left(\frac{f'g''-f''g'}{f'^3}\right)+Z_3^0K_3^2\frac{g'}{f'}+Z_3^0K_3^3g=0.
\end{aligned} \tag{4.43c}$$

Using the notations introduced in (4.30), relations (4.43) gives

$$\mathbb{Z}_3^1=-\frac{3f''}{f'}+\frac{3g'}{g}+\mathbb{K}_3^1f' \tag{4.44a}$$

$$\mathbb{Z}_3^2=-\frac{6f''g'}{f'g}+\frac{3f''^2}{f'^2}+\frac{3g''}{g}-\frac{f'''}{f'}+\mathbb{K}_3^1(\frac{2f'g'-f''g}{g})+\mathbb{K}_3^2f'^2 \tag{4.44b}$$

$$\begin{aligned}
\mathbb{Z}_3^3=&-\frac{3f''g''}{f'g}+\frac{3f''^2g'}{f'^2g}+\frac{g'''}{g}-\frac{f'''g'}{f'g}+\mathbb{K}_3^1\left(\frac{f'g''-f''g'}{g}\right)\\
&+\mathbb{K}_3^2\frac{f'^2g'}{g}+\mathbb{K}_3^3f'^3.
\end{aligned} \tag{4.44c}$$

4.3.3 $n = 4$**.** The expression of $\Psi^4 y$ and $\Phi^4 w$ are given by

$$K_4^0 y^{(4)} + K_4^1 y^{(3)} + K_4^2 y^{(2)} + K_4^3 y^{(1)} + K_4^4 y = 0, \tag{4.45}$$

$$Z_4^0 w^{(4)} + Z_4^1 w^{(3)} + Z_4^2 w^{(2)} + Z_4^3 w^{(1)} + Z_4^4 w = 0, \tag{4.46}$$

respectively. From equation (4.38), the expression of $y^{(4)}$ in terms of z and w are obtained as follows

$$y^{(4)} = \frac{dy^{(3)}}{dx} = \frac{dy^{(3)}}{dz}\frac{dz}{dx}. \tag{4.47}$$

Invoking (4.38), the above equation becomes

$$\begin{aligned} y^{(4)} =& \frac{d}{dz}\Big[\left(\frac{g}{f'^3}\right) w^{(3)} + \left(-\frac{3f''g}{f'^4} + \frac{3g'}{f'^3}\right) w^{(2)} + \Big(-\frac{6f''g'}{f'^4} + \frac{3f''^2 g}{f'^5} \\ &+ \frac{3g''}{f'^3} - \frac{f'''g}{f'^4}\Big) w' + \left(-\frac{3f''g''}{f'^4} + \frac{3f''^2 g'}{f'^5} + \frac{g'''}{f'^3} - \frac{f'''g'}{f'^4}\right) w\Big] \frac{dz}{dx}. \end{aligned} \tag{4.48}$$

We have,

$$
\begin{aligned}
y^{(4)} =& \frac{1}{f'}\Big(\frac{f'^3g' - 3f''f'^2g}{f'^6}w^{(3)} + \frac{g}{f'^3}w^{(4)} + \Big[\frac{3f'^3g'' - 3f''f'^2 3g'}{f'^6} \\
& - \frac{(3f'''g + 3f''g')f'^4 - 4f''f'^3 3f''g}{f'^8}\Big]w^{(2)} + (\frac{3g'}{f'^3} - \frac{3f''g}{f'^4})w^{(3)} + \\
& \Big[-\frac{(6f'''g' + 6f''g'')f'^4 - 24f''f'^3f''g'}{f'^8} \\
& + \frac{(6f'''f''g + 3f''^2g')f'^5 - 5f''f'^4.3f''^2g}{f'^{10}} + \frac{3f'^3g^{(3)} - 3f''f'^2.3g''}{f'^6} \\
& - \frac{(f^{(4)}g + f^{(3)}g')f'^4 - 4f''f'^3f^{(3)}g}{f'^8}\Big]w' + \Big(-\frac{6f''g'}{f'^4} + \frac{3f''^2g}{f'^5} \\
& + \frac{3g''}{f'^3} - \frac{f'''g}{f'4}\Big)w^{(2)} + \Big[-\frac{(3f^{(3)}g'' + 3f''g^{(3)})f'^4 - 4f''f'^3.3f''g''}{f'^8} \\
& + \frac{6f^{(3)}f''f'^5g' - 5f''f'^4.3f''^2g'}{f'^{10}} + \frac{g^{(4)}f'^3 - 3f''f'^2g^{(3)}}{f'^6} \\
& - \frac{(f^{(4)}g' + f^{(3)}g^{(2)})f'^4 - 4f''f'^3f^{(3)}g'}{f'^8}\Big]w + \Big(-\frac{3f''g''}{f'^4} + \frac{3f''^2g'}{f'^5} \\
& + \frac{g^{(3)}}{f'^3} - \frac{f^{(3)}g'}{f'^4}\Big)w'\Big)
\end{aligned} \tag{4.49a}
$$

$$
\begin{aligned}
=& \frac{g}{f'^4}w^{(4)} + \left(\frac{4g'}{f'^4} - \frac{6f''g}{f'^5}\right)w^{(3)} + \Big(\frac{6g''}{f'^4} - \frac{18f''g'}{f'^5} - \frac{4f^{(3)}g}{f'^5} \\
& + \frac{15f''^2g}{f'^6}\Big)w^{(2)} + \Big(-\frac{(8f^{(3)}g'}{f'^5} - \frac{18f''g''}{f'^5} + \frac{30f''^2g'}{f'^6} \\
& + \frac{10f^{(3)}f''g}{f'^6} - \frac{15f''^3g}{f'^7} + \frac{4g^{(3)}}{f'^4} - \frac{f^{(4)}g}{f'^5}\Big)w^{(1)} + \Big(\frac{g^{(4)}}{f'^4} - \frac{6f''g^{(3)}}{f'^5} \\
& - \frac{f^{(4)}g'}{f'^5} - \frac{4f^{(3)}g^{(2)}}{f'^5} + \frac{10f^{(2)}f^{(3)}g'}{f'^6} + \frac{12f''^2g''}{f'^6} - \frac{15f''^3g'}{f'^7}\Big)w
\end{aligned} \tag{4.49b}
$$

Substituting $y^{(4)}$, $y^{(3)}$, $y^{(2)}$, $y^{(1)}$ and y by their expression given respectively in (4.45), and taking into account the substitution $Z_4^0w^{(4)} = -(Z_4^1w^{(3)} + Z_4^2w^{(2)} + Z_4^3w^{(1)} + Z_4^4w)$, we have,

$$
\begin{aligned}
& - K_4^0\alpha_{41}(Z_4^1w^{(3)} + Z_4^2w^{(2)} + Z_4^3w^{(1)} + Z_4^4w) \\
& + (K_4^0Z_4^0\alpha_{42} + Z_4^0K_4^1\alpha_{31})w^{(3)} \\
& + (Z_4^0K_4^0\alpha_{43} + Z_4^0K_4^1\alpha_{32} + Z_4^0K_4^2\alpha_{21})w^{(2)} \\
& + (Z_4^0K_4^0\alpha_{44} + Z_4^0K_4^1\alpha_{33} + Z_4^0K_4^2\alpha_{22} + Z_4^0K_4^3\alpha_{11})w^{(1)} \\
& + (Z_4^0K_4^0\alpha_{45} + Z_4^0K_4^1\alpha_{34} + Z_4^0K_4^2\alpha_{23} + Z_4^0K_4^3\alpha_{12} + Z_4^0K_4^4g)w = 0,
\end{aligned} \tag{4.50}
$$

where

$$
\begin{aligned}
&\alpha_{11} = \frac{g'}{f'}, \quad \alpha_{12} = \frac{g}{f'}, \quad \alpha_{21} = \frac{g}{f'^2}, \quad \alpha_{22} = \frac{2f'g' - f''g}{f'^3}, \quad \alpha_{23} = \frac{f'g'' - f''g'}{f'^3}, \\
&\alpha_{31} = \frac{g}{f'^3}, \quad \alpha_{32} = -\frac{3f''g}{f'^4} + \frac{3g'}{f'^3}, \quad \alpha_{33} = -\frac{6f''g'}{f'^4} + \frac{3f''^2 g}{f'^5} + \frac{3g''}{f'^3} - \frac{f'''g}{f'^4} \\
&\alpha_{34} = -\frac{3f''g''}{f'^4} + \frac{3f''^2 g'}{f'^5} + \frac{g'''}{f'^3} - \frac{f'''g'}{f'^4}, \quad \alpha_{41} = \frac{g}{f'^4}, \quad \alpha_{42} = \frac{4g'}{f'^4} - \frac{6f''g}{f'^5}, \\
&\alpha_{43} = \frac{6g''}{f'^4} - \frac{18f''g'}{f'^5} - \frac{4f^{(3)}g}{f'^5} + \frac{15f''^2 g}{f'^6} \\
&\alpha_{44} = -\frac{(8f^{(3)}g'}{f'^5} - \frac{18f''g''}{f'^5} + \frac{30f''^2 g'}{f'^6} + \frac{10f^{(3)}f''g}{f'^6} - \frac{15f''^3 g}{f'^7} + \frac{4g^{(3)}}{f'^4} - \frac{f^{(4)}g}{f'^5} \\
&\alpha_{45} = \frac{g^{(4)}}{f'^4} - \frac{6f''g^{(3)}}{f'^5} - \frac{f^{(4)}g'}{f'^5} - \frac{4f^{(3)}g^{(2)}}{f'^5} + \frac{10f^{(2)}f^{(3)}g'}{f'^6} + \frac{12f''^2 g''}{f'^6} - \frac{15f''^3 g'}{f'^7}.
\end{aligned} \tag{4.51}
$$

Equation (4.50) can be written as

$$
\begin{aligned}
&(-K_4^0 Z_4^1 + Z_4^0 K_4^0 \alpha_{42} + Z_4^0 K_4^1 \alpha_{31}) w^{(3)} \\
&+ (-Z_4^2 K_4^0 \alpha_{41} + Z_4^0 K_4^0 \alpha_{43} + Z_4^0 K_4^1 \alpha_{32} + Z_4^0 K_4^2 \alpha_{21}) w^{(2)} \\
&+ (-K_4^0 Z_4^3 K_4^0 \alpha_{41} + Z_4^0 K_4^0 \alpha_{44} + Z_4^0 K_4^1 \alpha_{33} + Z_4^0 K_4^2 \alpha_{22} + Z_4^0 K_4^3 \alpha_{11}) w^{(1)} \\
&+ (-K_4^0 Z_4^4 \alpha_{41} + Z_4^0 K_4^0 \alpha_{45} + Z_4^0 K_4^1 \alpha_{34} + Z_4^0 K_4^2 \alpha_{23} \\
&+ Z_4^0 K_4^3 \alpha_{12} + Z_4^0 K_4^4 g) w = 0.
\end{aligned} \tag{4.52}
$$

Again, we can equate the coefficients of w and its derivatives to zero to get

$$\mathbb{Z}_4^1 = f'\mathbb{K}_4^1 + \frac{4g'}{g} - \frac{6f''}{f'}, \tag{4.53a}$$

$$\mathbb{Z}_4^2 = f'^2\mathbb{K}_4^2 + (\frac{3f'g'}{g} - 3f'')\mathbb{K}_4^1 + \frac{6g''}{g} - \frac{18f''g'}{f'g} - \frac{4f^{(3)}}{f'} + \frac{15f''^2}{f'^2}, \tag{4.53b}$$

$$\begin{aligned}\mathbb{Z}_4^3 =& f'^3\mathbb{K}_4^3 + (\frac{2f'^2g'}{g} - f''f')\mathbb{K}_4^2 + (-\frac{6f''g'}{g} + \frac{3f''^2}{f'} + \frac{3f'g''}{g} \\ & - f^{(3)})\mathbb{K}_4^1 - \frac{8f^{(3)}g'}{f'g} - \frac{18f''g''}{f'g} + \frac{30f''^2g'}{f'^2g} + \frac{10f^{(3)}f''}{f'^2} - \frac{15f''^3}{f'^3} \\ & + \frac{4g^{(3)}}{g} - \frac{f^{(4)}}{f'},\end{aligned} \tag{4.53c}$$

$$\begin{aligned}\mathbb{Z}_4^4 =& f'^4\mathbb{K}_4^4 + \frac{f'^3g'}{g}\mathbb{K}_4^3 + (\frac{f'^2g''}{g} - \frac{f''f'g'}{g})\mathbb{K}_4^2 + (-\frac{3f''g''}{g} + \frac{3f''^2g'}{f'g} \\ & + \frac{f'^2g^{(3)}}{g} - \frac{f^{(3)}g'}{g})\mathbb{K}_4^1 + \frac{g^{(4)}}{g} - \frac{6f^{(2)}g^{(3)}}{f'g} - \frac{f^{(4)}g'}{f'g} - \frac{4f^{(3)}g^{(2)}}{f'g} \\ & + \frac{10f^{(3)}f''g'}{f'^2g} + \frac{12f''^2g''}{f'^2g} - \frac{15f''^3g'}{f'^3g}.\end{aligned} \tag{4.53d}$$

(4.53e)

In (4.13) we have $y = y(x)$, $K_n^i = K_n^i(x)$, $i = 0, \ldots, n$ and similarly in (4.19) $w = w(z)$, and $Z_n^i = Z_n^i(x)$, for $j = 0, \ldots, n$. Now we have established a general relationship between the A_n^j and B_n^j, i.e

$$A_n^j = \mathbb{K}_n^j|_{\mathbb{K}_n^1=0} \quad \text{and} \quad B_n^j = \mathbb{Z}_n^j|_{\mathbb{Z}_n^1=0}. \tag{4.54}$$

4.3.4 Example. For $n = 2$, the transformation (4.11) implies that $g = \lambda\sqrt{f'}$. Therefore, expression (4.31) becomes

$$\mathbb{Z}_2^1 = f'\mathbb{K}_2^1 + \frac{\lambda f''f'^{\frac{1}{2}} - \lambda f''f'^{\frac{1}{2}}}{\lambda f'^{\frac{3}{2}}} \tag{4.55a}$$

$$\mathbb{Z}_2^2 = f'^2\mathbb{K}_2^2 + \frac{\frac{\lambda}{2}f''f'^{\frac{1}{2}}}{\lambda f'^{\frac{1}{2}}}\mathbb{K}_2^1 + \frac{\frac{\lambda}{2}f^{(3)}f'^{\frac{1}{2}} - \frac{\lambda}{4}f''^2f'^{-\frac{1}{2}} - \frac{\lambda}{2}f''^2f'^{-\frac{1}{2}}}{\lambda f'^{\frac{3}{2}}}. \tag{4.55b}$$

Letting $\mathbb{K}_2^1 = 0$ and using the notation (4.54) we obtain $\mathbb{Z}_2^1 = 0$, as expected, and

$$B_2^2 = \frac{1}{f'^2}\left[A_2^2 f'^4 - \frac{3}{4}f''^2 + \frac{1}{2}f'f^{(3)}\right]. \tag{4.56}$$

We now move on to find an expression for R in terms of r. To do so we may assume that the equation is in its reduced form, which also assumes the equality $S = -(n-1)R'/2$.

For simplicity, but without loss of generality, we may assume that the equations are in their reduced normal form (4.14) and (4.20). As already mentioned, suppose that the parameter of the source equation generating (4.14) is $r = r(x)$. Given that equivalent equations have isomorphic symmetry algebras, equation (4.20) is also iterative and we wish to find the corresponding parameter $R = R(z)$ of its source equation. We also need to recall that by considering the equation to be in its normal form, the point transformations (4.4) reduces to (4.11), i.e.

$$x = f(z), \qquad y = \lambda\,[f'(z)]^{\frac{n-1}{2}}\, w. \tag{4.57}$$

A direct calculation (we let $g = [f'(z)]^{\frac{n-1}{2}}\, w$ and $B_n^i = \mathbb{Z}_n^i|_{\mathbb{Z}_n^1=0}$ in the above calculation) shows that in terms of the parameters λ and f of the equivalence transformation and the coefficients A_n^i of the original equation, we have

$$B_2^2 = \frac{1}{f'^2}\left[A_2^2\, f'^4 - \frac{3}{4} f''^2 + \frac{1}{2} f' f^{(3)}\right], \qquad \textit{for} \quad n = 2 \tag{4.58}$$

$$B_3^2 = \frac{1}{f'^2}\left[A_3^2\, f'^4 - 3 f''^2 + 2 f' f^{(3)}\right], \qquad \textit{for} \quad n = 3 \tag{4.59}$$

$$B_4^2 = \frac{1}{2f'^2}\left[2A_4^2\, f'^4 - 15 f''^2 + 10 f' f^{(3)}\right], \qquad \textit{for} \quad n = 4. \tag{4.60}$$

On the other hand we know that by assuming r and R to be the parameters of the source equations for (4.14) and (4.20) respectively, we have for $n \geq 2$

$$A_n^2(x) = \binom{n+1}{3} A(r) \tag{4.61a}$$

$$B_n^2(z) = \binom{n+1}{3} A(R), \tag{4.61b}$$

where

$$A\left(r(x)\right) = \frac{r'^2 - 2rr''}{4r^2}. \tag{4.61c}$$

Consequently substituting the above expressions for A_n^2 and B_n^2 in terms of r and R respectively in (4.59) would yield the determining equation for R when $n = 3$. Namely, we have

$$\frac{R'^2 - 2RR''}{R^2} = \frac{1}{f'^2}\left[\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 3 f''^2 + 2 f' f^{(3)}\right], \tag{4.62}$$

where $f = f(z)$. Similarly, for $n = 4$, the determining equation for R takes the form

$$\frac{10}{4}\frac{R'^2 - 2RR''}{R^2} = \frac{1}{2f'^2}\left[\frac{10}{4} \cdot 2\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 15 f''^2 + 10 f' f^{(3)}\right] \tag{4.63}$$

which is equivalent to

$$\frac{R'^2 - 2RR''}{R^2} = \frac{1}{f'^2}\left[\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2} f'^4 - 3 f''^2 + 2 f' f^{(3)}\right]. \tag{4.64}$$

As it should be expected, equations (4.62) and (4.64) are the same and correspond to that derived from (4.58), which is due to the fact that in reality the expression for R does not depend on the order of the equation. In order word, we only need to know this expression for the second-order source equation.

Note that equation (4.62) has the form

$$\frac{R'^2 - 2RR''}{R^2} = H(z), \tag{4.65}$$

where H is a given function. Therefore, if we let r or f be arbitrary functions, we may not be able to solve (4.62) for R, because the solution of the differential equation (4.65) is not available for B_2^2 an arbitrary function. Classes of solution to equation (4.62) have been obtained in chapter 3.

4.3.5 Example. Let us illustrate our work in the case where A_2^2 equals a constant. We then have

$$\frac{r'^2 - 2rr''}{4r^2} = C \tag{4.66}$$

and we know from the previous discussion that the expression for r corresponding to (4.66) is

$$r = c_2 \cos^2\left[\sqrt{C}(c_1 + x)\right], \tag{4.67}$$

where c_1 and c_2 are arbitrary constants. This r is the parameter of the source equation generating a linear iterative equation with a non-zero constant coefficient. If in (4.58) we suppose that $f(z) = \alpha z + \beta$, where $\alpha \neq 0$ and β are given constants, we get

$$\frac{R'^2 - 2RR''}{4R^2} = C\,\alpha^2. \tag{4.68}$$

Therefore the value of R is given by

$$R = c_2 \cos^2\left[\alpha\sqrt{C}(c_1 + x)\right]. \tag{4.69}$$

4.4 Conclusion

We performed the point transformations one can use to retrieve other solutions of the linear iterative equation from the existing solutions. We then used the transformation (4.57) to generate the parameter of the transformed equation under equivalence transformation. We noticed that the calculations for $n = 2, 3, 4$ lead to the relationship between the parameter of the original equation and the one of the transformed equation under equivalence transformation. This relationship does not depend on the order of the linear iterative equation.

5. Conservation Laws

5.1 Introduction

In this chapter we concentrate on finding conservation laws of the second-order source equation. In Chapter 3, we gave the eight symmetries of the equation and given that symmetry group of Euler-Lagrange equations give rise to a conservation law, we aim to use some symmetries (Noether's symmetries) of the corresponding Euler-Lagrange equations to find the conserved quantities of (1.14). The method of mapping the second-order source equation to the canonical form $Y'' = 0$ has been considered and some special cases have been studied.

5.2 Lagrangian, Noether symmetries and conserved quantities.

The calculus of variations is a generalization of the problem of optimization of functions depending on functionals. Given that a differentiable functional is stationary at its local extrema, we want to introduce the Euler-Lagrange equation whose solutions are the functions for which the functional is stationary.

5.2.1 Definition. For $1 \leq \alpha \leq q$, the α-th *Euler operator* is given by

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \tag{5.1}$$

the sum extending over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$, $k \geq 0$.

The Euler-Lagrange operator is useful tool in the calculus of variations.

5.2.2 Theorem. *If $u = f(x)$ is a smooth extremal of the variational problem $\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx$, then it must be a solution of the Euleur- Lagrange equations*

$$E_\mu(L), \qquad \mu = 1, \ldots, q. \tag{5.2}$$

5.2.3 Lagrangian of the second-order source equation. Consider our second-order source equation

$$y'' + A_2^2(x)y = 0. \tag{5.3}$$

In this special case, the Euler-Lagrange equation is given by

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} =0, \tag{5.4}$$

ie

$$y'' = \frac{1}{\frac{\partial^2 L}{\partial y'^2}} \left(\frac{\partial L}{\partial y} - y' \frac{\partial^2 L}{\partial y \partial y'} - \frac{\partial^2 L}{\partial y' \partial x} \right). \tag{5.5}$$

If we assume that

$$L = \frac{1}{2} g(x) y'^2 + h(x,y) \tag{5.6}$$

then the equation (5.5) , taking into consideration $y'' = -A_2^2 y$, becomes

$$\frac{1}{g(x)} \left(h_y - g_x y' \right) = - A_2^2 y. \tag{5.7}$$

Separating by $(y')^n$ we get

$$y' : \frac{g_x}{g} = 0 \tag{5.8a}$$

$$1 : h_y = -g A_2^2 y. \tag{5.8b}$$

Equation (5.8) implies that

$$g(x) = K, \qquad h(x,y) = -\frac{1}{2} g A_2^2 y^2 + \phi(x). \tag{5.9}$$

Therefore, the Lagrangian takes the form

$$L = \frac{1}{2} K y'^2 - \frac{1}{2} K\, A_2^2\, y^2 + \phi(x), \tag{5.10}$$

where K is a constant and ϕ is an arbitrary function. By letting $\phi = 0$, the Lagrangian of the the second-order source equation is given by

$$L = \frac{1}{2} y'^2 - \frac{1}{2} A_2^2\, y^2. \tag{5.11}$$

5.2.4 Noether symmetries of the second-order source equation. Let $v = \xi(x,y)\partial_x + \phi(x,y)\partial_y$, the first prolongation of v is given by

$$X = \xi \partial_x + \phi \partial_y + (\phi_x + \phi_y y' - \xi_x y' - \xi_y y'^2)\partial_{y'}. \tag{5.12}$$

We know that X leaves invariant the functional in the calculus of variations up to 'the gauge term $F(x,y)$' if

$$\int L(x^\star, y^\star, y^{\star\prime}) dx^\star = \int L(x,y,y') dx + \varepsilon F(x,y) \tag{5.13}$$

i.e. if

$$X\, L + L \frac{d\xi}{dx} = \frac{dF}{dx}. \tag{5.14}$$

The substitution of L and X with their expression, given in (5.12) and (5.14), into (5.11) leads to

$$\left(\phi_x + \phi_y y' - \xi_x y' - \xi_y y'^2\right) y' + \phi(-A_2^2 y) - \frac{1}{2}\xi A_2^{2\prime} y^2 + \frac{1}{2}\xi_x y'^2 - \frac{1}{2} A_2^2 \xi_x y^2 = F_x + y' F_y. \quad (5.15)$$

Separating by the powers of derivatives of y we get

$$1 : -A_2^2 \phi y - \frac{1}{2} A_2^2 \xi_x y^2 - \frac{1}{2}\xi A_2^{2\prime} y^2 = F_x, \quad (5.16a)$$
$$y' : \phi_x = F_y, \quad (5.16b)$$
$$y'^2 : \phi_y - \xi_x + \frac{1}{2}\xi_x = 0, \quad (5.16c)$$
$$y'^3 : -\xi_y = 0. \quad (5.16d)$$

From (5.16d), i.e. $\xi_y = 0$, we get

$$\xi = f(x), \quad (5.17)$$

where f is an arbitrary function. Then now from (5.16c) we get

$$\phi_y = \frac{1}{2}\xi_x \quad (5.18)$$
$$= \frac{1}{2} f'(x), \quad (5.19)$$

which implies that

$$\phi(x, y) = \frac{1}{2} f'(x) y + g(x). \quad (5.20)$$

Integrating (5.16b), ie $F_y = \phi_x$, with respect to y (taking into consideration (5.20)) we get

$$F(x, y) = \frac{1}{4} f''(x) y^2 + g'(x) y + h(x). \quad (5.21)$$

We now use (5.17), (5.20) and (5.21) in (5.16a) and the equations obtained by the separation of the monomials in the the resulting equation are given as follow

$$1 : h' = 0, \quad (5.22a)$$
$$y : A_2^2 g = g^{(2)}, \quad (5.22b)$$
$$y^2 : -\frac{1}{2} A_2^2 f' - \frac{1}{2} A_2^2 f' - \frac{1}{2} A_2^{2\prime} f = \frac{1}{4} f^{(3)}. \quad (5.22c)$$

The above equations are equivalent to

$$h = c_6, \quad (5.23a)$$
$$g^{(2)} + A_2^2 g = 0, \quad (5.23b)$$
$$f^{(3)} + 4\, A_2^2 f' + 2\, A_2^{2\prime} f = 0. \quad (5.23c)$$

Equations (5.23b) and (5.23c) are ordinary iterative equations with the same source equation $y'' + A_2^2 y = 0$. Therefore their solutions (according to results (3.36)) are given by

$$g(x) = c_4 u + c_5 v, \tag{5.24}$$
$$f(x) = c_1 u^2 + c_2 uv + c_3 v^2, \tag{5.25}$$

respectively, where u and v solutions of $y'' + A_2^2 = 0$. So by letting $h = 0$ we have

$$v = \left(c_1 u^2 + c_2 uv + c_3 v^2\right) \partial_x + \left[\left(c_1 uu' + \frac{1}{2} c_2 (u'v + uv') + c_3 vv'\right) y + c_4 u + c_5 v\right] \partial_y \tag{5.26}$$

and

$$F(x, y) = \left[\frac{1}{2} c_1 (u'^2 + uu'') + \frac{1}{4} c_2 (u''v + 2u'v' + uv'') + \frac{1}{2} c_3 (v'^2 + vv'')\right] y^2 + (c_4 u + c_5 v') y. \tag{5.27}$$

Taking respectively c_1, c_2, c_3, c_4, c_5 to one and the remaining constants to zero allows us to find the basis as below

$$v_1 = u^2 \partial_x + uu' y \partial_y; \quad F_1 = \frac{1}{2} (u'^2 + uu'') y^2 \tag{5.28a}$$
$$v_2 = uv \partial_x + \frac{1}{2} (u'v + uv') y \partial_y; \quad F_2 = \frac{1}{4} (u''v + 2u'v' + uv'') y^2 \tag{5.28b}$$
$$v_3 = v^2 \partial_x + vv' y \partial_y; \quad F_3 = \frac{1}{2} (v'^2 + vv'') y^2 \tag{5.28c}$$
$$v_4 = u \partial_y; \quad F_4 = u' y \tag{5.28d}$$
$$v_5 = v \partial_y; \quad F_5 = v' y. \tag{5.28e}$$

The above symmetries are the five Noether symmetries of the second-order source equation $y'' + A_2^2\, y = 0$.

5.2.5 Conserved quantities of the second-order source equation. To find the conserved quantities using the Noether symmetries we invoke the following theorem

5.2.6 Theorem. *If X is a Noether symmetry, F a corresponding function for a lagrangian L, then*

$$I = L\xi + (\phi - y'\xi) \frac{\partial L}{\partial y'} - F \tag{5.29}$$

is a conserved quantity.

It follows from the previous results and the above theorem that the conserved quantities of the second-order source equation $y'' + A_2^2 y = 0$ are given by

$$I_1 = uu'yy' - \frac{1}{2} u^2 y'^2 - \frac{1}{2} u'^2 y^2, \tag{5.30a}$$
$$I_2 = \frac{1}{2} (u'v + uv') yy' - \frac{1}{2} uvy'^2 - \frac{1}{2} u'v'y^2, \tag{5.30b}$$
$$I_3 = vv'yy' - \frac{1}{2} v^2 y'^2 - \frac{1}{2} v'^2 y^2, \tag{5.30c}$$
$$I_4 = uy' - u'y, \tag{5.30d}$$
$$I_5 = vy' - v'y. \tag{5.30e}$$

5.2.7 Equivalent Lagrangians. By definition [17, 18, 19], two Lagrangians $L(x, y, y')$ and $\overline{L}(X, Y, Y')$ are said to be equivalent up to gauge $F = F(x, y)$ if

$$L(x, y, y') = \overline{L}(X, Y, Y')\frac{dX}{dx} + \frac{dF}{dx} \tag{5.31}$$

where $X = X(x, y)$ and $Y = Y(x, y)$. We are going to use this definition to reduce (5.3), ie $y'' + A_2^2 y = 0$, to the canonical form

$$Y'' = 0. \tag{5.32}$$

From (5.11) we know that the Lagragian of $y''+A_2^2 y = 0$ is $L = \frac{1}{2}y'^2 - \frac{1}{2}A_2^2 y^2$, and the Lagrangian of $Y'' = 0$ is $\overline{L} = \frac{1}{2}Y'^2$. Let us find $X = X(x, y)$ and $Y = Y(x, y)$ such that L and $\overline{L}$ equivalent up to $F = F(x, y)$ by solving

$$\begin{aligned}\frac{1}{2}y'^2 - \frac{1}{2}A_2^2 y^2 =& \frac{1}{2}Y'^2\frac{dX}{dx} + \frac{dF}{dx}\\ =& \frac{1}{2}\left(\frac{dY}{dx}\frac{dx}{dX}\right)^2\frac{dX}{dx} + \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}\\ =& \frac{(Y_x + y'Y_y)^2}{2\,(X_x + y'X_y)} + F_x + y'F_y.\end{aligned} \tag{5.33}$$

Rearranging the above equation we get

$$(X_x + y'X_y)(y'^2 - A_2^2 y^2) = (Y_x + y'Y_y)^2 + 2(F_x + y'F_y)(X_x + y'X_y). \tag{5.34}$$

The separation by y'^n leads to

$$1 : X_x A_2^2 y^2 + Y_x^2 + 2F_x\xi_x = 0, \tag{5.35a}$$
$$y' : -A_2^2 X_y y^2 - 2Y_x Y_y + X_x F_y - 2F_x X_y = 0, \tag{5.35b}$$
$$y'^2 : X_x - Y_y^2 - 2X_y F_y = 0, \tag{5.35c}$$
$$y'^3 : X_y = 0. \tag{5.35d}$$

From (5.35d), ie $X_y = 0$, we deduce that X is independent of y and then the above governing equations are reduced to

$$X_x A_2^2 y^2 + Y_x^2 + 2X_x F_x = 0, \tag{5.36a}$$
$$Y_x = \sqrt{X_x}, \tag{5.36b}$$
$$Y_x Y_y + F_y X_x = 0. \tag{5.36c}$$

Let $X_x = b(x)^2$ for some function i of x. From (5.36b), i.e. $Y_x = \sqrt{X_x}$, we get

$$Y = b(x)y + c_1(x) \tag{5.37a}$$

for some function c_1 depending on x only. From (5.36c), i.e. $X_x - Y_y^2 - 2X_y F_y = 0$, we get

$$F = -\frac{1}{2}\frac{b'}{b}y^2 - \frac{c_1'}{b}y + c_2(x), \tag{5.37b}$$

where c_2 is an arbitrary function of x. The substitution of F and Y with their expressions given in (5.37) into (5.36a) yields

$$\left[b^2 A_2^2 + b'^2 - b^2 \left(\frac{b'}{b}\right)'\right] y^2 + \left[2b'c_1' - 2b^2 \left(\frac{c_1'}{b}\right)'\right] y + c_1'^2 + 2b^2 c_2' = 0. \tag{5.38}$$

The separation of the monomials gives

$$A_2^2 + \left(\frac{b'}{b}\right)^2 - \left(\frac{b'}{b}\right)' = 0, \tag{5.39a}$$

$$b'c_1' - b^2 \left(\frac{c_1'}{b}\right)' = 0, \tag{5.39b}$$

$$c_1'^2 + 2b^2 c_2' = 0. \tag{5.39c}$$

The functions c_1, c_2 and b (thus X, Y and F) are known once we solve (5.39a), i.e.

$$A_2^2 + B^2 = B', \tag{5.40}$$

where $B = \dfrac{b'}{b}$.

For example, if A_2^2 equals to a constant c then we can let

$$B(x) = \sqrt{c}\tan\left(\sqrt{c}x\right), \quad b(x) = \sec\left(\sqrt{c}x\right), \quad c_1 = 0, \quad c_2 = 0 \tag{5.41}$$

to get

$$X = \frac{1}{\sqrt{c}}\tan\left(\sqrt{c}x\right), \quad Y = \sec\left(\sqrt{c}x\right), \quad F = -\sqrt{c}\tan\left(\sqrt{c}x\right)\sec\left(\sqrt{c}x\right) y. \tag{5.42}$$

5.3 Conclusion

In this chapter we determined the general form of the Lagrangian of the second-order source equation. Using the expression obtained for the Lagrangian, we found the five Noether symmetries (with the corresponding 'gauge functions') and the conserved quantities of the second-order equation. We also tried the mapping of the second-order source equation onto its canonical form. In the process of finding the mapping, we encounter equation (5.40) which is not solvable in general.

6. Conclusion

In this dissertation, we have exploited known results about the symmetry and superposition properties of linear iterative equations to obtain new solutions of these equations and to express them in terms of the parameters of the source equation. This has yielded an expression of its symmetries in terms of the parameters r and s of the first-order source equation for some functions. We have obtained some properties of these equations and, in particular, we have derived expressions for the parameters of the source equation under equivalence transformations, and we have derived some conservation laws of the second-order source equation.

In Chapter 3, we have reviewed the results obtained by Krause and Michel [1], i.e. the expression of the symmetry generator of the linear iterative equation in terms of the solutions of the second-order source equation. We have obtained their results by a slightly different method which consists of substituting (3.52) into (3.34). We made use of the expression (1.10) to reduce the condition on the infinitesimal ξ. We have proved that the condition on the infinitesimal function $\xi = f(x)$ does not depend on the order of the linear iterative equation. Based on some properties of a linear iterative equation, we obtained some solutions of the linear iterative equation of a general order and their expressions in terms of the parameters r, s of the first-order source equation. We also gave a list of linear iterative equations with constant coefficients expressed in terms of the coefficient A_2^2.

In Chapters 4 and 5, some results concerning the parameters of the transformed equation under equivalence transformation were obtained for the linear iterative equation of order n, and finally we have derived the Lagrangian and we gave the Noether symmetries of the second-order source equation. We have used these symmetries together with the Lagrangian to get the conserved quantities and to map the source equation to its canonical form. It is worthwhile to mention that the reduction of the second-order source equation using the symmetry $v_0 = y\partial_y$ (see section 3.4.7) and the method of equivalent Lagrangians lead to the same equation (see (3.162) and (5.40)).

References

[1] J. Krause, L. Michel, Equations différentielles linéaires d'ordre $n > 2$ ayant une algèbre de Lie de symétrie de dimension $n + 4$, C.R. Acad. Sci. Paris 307 (1988) 905–910.

[2] S. Lie, Klassification und Integration von gewöhnlichen Differentialgleichungen zwischen $x, y,$ die eine Gruppe von Transformationen gestetten. I, Math. Ann. 22 (1888) 213–253.

[3] F. M. Mahomed, Symmetry group classification of ordinary differential equations: Survey of some results, Math. Meth. Appl. Sci. 20 (2007) 1995–2012.

[4] J.C Ndogmo and F.M. Mahomed, A characterization of iterative equations by their coefficients, prepint.

[5] Olver, P. J. : Applications of Lie Groups to Differential Equations. Springer, New York (1986).

[6] Olver, P. J. : Equivalence, Invariants, and Symmetry. Cambridge University Press, Cambridge (1995) equations by their coefficients, preprint.

[7] Winternitz, P. : Lie groups and solutions of nonlinear partial differential equations. Preprint CRM-1841 (1993).

[8] Oliveri, Francesco Lie symmetries of differential equations: classical results and recent cvontributions, Symmetry 2 (2010), no. 2, 658706.

[9] J.C. Ndogmo, Symmetry properties of a nonlinear acoustics model, Nonlinear Dyn, 55 (2009) 151-167.

[10] Senthilvelan, M. ; Torrisi, M. Equivalence transformations and approximate solutions of a nonlinear heat conduction model. J. Phys. A 31 (1998), no. 49, 1000510016.

[11] Ibragimov, R. N. Approximation equivalence transformations and invariant solutions of a perturbed nonlinear wave equation. Electron. J. Differential Equations 2001, No. 23, 12 pp.

[12] J.C. Ndogmo, A method for the equivalence group and its infinitesimal generators, J. Phys. A: Math. Theo. 41 (2008) 102001 (A Fast Track Communication).

[13] F.M. Mahomed, Symmetry Lie algebra of the nth ordinary differential equations.

[14] E. Laguerre, Sur quelques invariants des équations différentielles linéaires, C.R. Acad. Sci. Paris 88 (1879) 224-227.

[15] P.J. O'Hara, R. Osteen, and R.S. Rodriguez, " A method of solving $y^{(k)} - f(x)y = 0$", *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 1, pp. 183-188, 1992.

[16] A.H. Kara and F.M. Mahomed, Relationship between symmetries and conservation laws, *Int, J. Theoretical Physics*, Vol.39, No. 1, 2000, 23-40.

[17] A.H. Kara and N. Wilson, "Equivalent Lagrangians: Generalization, Transformation Maps, and Applications", *Journal of Applied Mathematics*, Vol. 2012, Article ID 860482, 19 pages.

[18] A. H. Kara, "Equivalent lagrangians and the inverse variational problem with applications", *Quaestiones Mathematicae*, Vol. 27, no. 2, pp. 207-216, 2004.

[19] A. H. Kara and F. M. Mahomed, "Equivalent lagrangians and the solution of some classes of nonlinear equations $\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2q^{-1} + f(t)q^n$", *International Journal of Non-Linear Mechanics*, Vol. 27, no. 6, pp. 919-927, 1992.